Lesson 7: Similar Polygons

Goals

- Comprehend the phrase “similar polygons” (in written and spoken language) to mean the polygons have congruent corresponding angles and proportional side lengths.
- Critique (orally) arguments that claim two polygons are similar.
- Justify (orally) the similarity of two polygons given their angle measures and side lengths.

Learning Targets

- I can use angle measures and side lengths to conclude that two polygons are not similar.
- I know the relationship between angle measures and side lengths in similar polygons.

Lesson Narrative

One of the powerful things about the definition of similarity in terms of transformations is that we can talk about whether two figures are similar even when they are not composed of straight lines. For example, we can show that all circles are similar, because we can translate one so they have the same center and then dilate one until it matches the other.

In the case of polygons, we can understand similarity by examining side lengths and angle measures. Since the transformations we have studied (translations, rotations, reflections, dilations) do not change angle measures, similar polygons have congruent corresponding angles. Only dilations change side lengths and they change them all by the *same* scale factor. This means that similar polygons have proportional corresponding side lengths. In general, both side lengths and angle measures are important to determine whether or not two polygons are similar. The next lesson will examine the special case of triangles where it turns out that congruent corresponding angles is all that is needed to conclude that two triangles are similar.

The focus in this lesson is on quadrilaterals, and students determine efficient ways to decide whether certain types of quadrilaterals are similar:

- Two rectangles are similar if the side lengths are proportional.
- Two rhombuses are similar if the angles are congruent.
- All squares are similar.

Alignments

Addressing

- 8.G.A.2: Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

- 8.G.A.4: Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR2: Collect and Display

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Pre-printed slips, cut from copies of the blackline master

Required Preparation

Make 1 copy of the blackline master for every 10 students, and cut them up ahead of time.

> ### Student Learning Goals
> Let’s look at sides and angles of similar polygons.

7.1 All, Some, None: Congruence and Similarity

Warm Up: 10 minutes
Students examine statements deciding in each case whether the statement is always true, sometimes true, or never true. Since figures are similar if one is the result of reflecting, rotating, translating, and dilating the other, and figures are congruent only as a result of the rigid transformations, congruent figures are similar but not vice versa. Though students have not studied how angles behave under dilations, students have had multiple opportunities to see that dilations influence the size of a shape but not its angles. They also know from work in grade 7 that scaled copies have corresponding angles with the same measure, and dilations have been characterized as a process that makes scaled copies.

Addressing

- 8.G.A.2
- 8.G.A.4

Launch

Display all 3 statements. Ask students to decide whether each of the statements is always, sometimes, or never true and give a signal when they have reasoning to support their decision.

> ### Student Task Statement
>
> Choose whether each of the statements is true in *all* cases, in *some* cases, or in *no* cases.
>
> 1. If two figures are congruent, then they are similar.
> 2. If two figures are similar, then they are congruent.
> 3. If an angle is dilated with the center of dilation at its vertex, the angle measure may change.

Student Response

1. All cases. Congruent figures can be taken from one to the other by using translations, rotations, and reflections. Similar figures use these same transformations, but also use dilations. If we don’t do a dilation, the figures are still similar.
2. Some cases. Similar figures can be taken from one to the other using translations, rotations, reflections, and dilations. Congruent figures do not allow dilations. Two figures can be similar that are different sizes (a square with side length one and a square with side length two). These figures are not congruent.
3. No cases. The dilation will take the rays of the angle to themselves and so will not change the measure of the angle.

Activity Synthesis

Discuss each statement one at a time with this structure:

- Poll the class on their answer choice and display the answers.
- If everyone agrees on one answer, ask a few students to share their reasoning, recording it for all to see.
- If there is disagreement on an answer, ask students with opposing answers to explain their reasoning to come to an agreement on an answer.

7.2 Are They Similar?

10 minutes

In the previous lesson, students saw that figures are similar when there is a sequence of translations, rotations, reflections, and dilations that map one figure onto the other. This activity focuses on some common misconceptions about similar figures, and students have an opportunity to critique the reasoning of others (MP3). Two polygons with proportional side lengths but different angles are not similar and two polygons with the same angles but side lengths that are not proportional are also not similar. Reasoning through these problems will help foster student thinking about when two polygons are *not* similar.

Monitor for students who formulate these ideas for establishing that two figures are *not* similar:

- The angle measures are different (problem 1).
- The side lengths need to be multiplied by *different* scale factors in order to match up.

Select these students to share during the discussion.

Addressing

- 8.G.A.4

Instructional Routines

- MLR1: Stronger and Clearer Each Time

Launch

Tell students they will look at some polygons that are claimed to be similar. What are some characteristics of similar polygons that are easy to recognize that can be used to confirm or deny the claims? Give 3 minutes of quiet work time followed by a whole-class discussion.

Access for Students with Disabilities

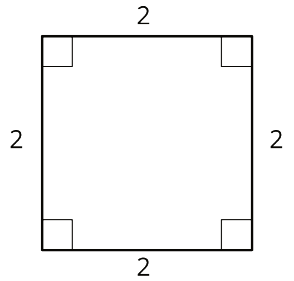
Action and Expression: Develop Expression and Communication. To help get students started, display sentence frames such as: “I agree with ___ because….” Encourage students to use the side lengths, angle measures, and scale factor in their reasoning.
Supports accessibility for: Language; Organization

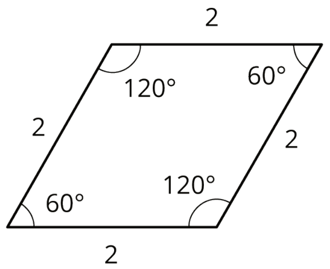
Anticipated Misconceptions

Students might think that checking for angle congruence alone can determine similarity. They might also think that just checking if all side lengths have the same scaled value will be enough to determine similarity. However, both the angle sizes and the scale factors must be checked together in determining similarity.

Students might think the side lengths must be different in order for two figures to be similar, but a dilation does not need to be used in the sequence of transformations to show similarity (recall the warm-up where students saw that congruent figures are always similar).

Student Task Statement

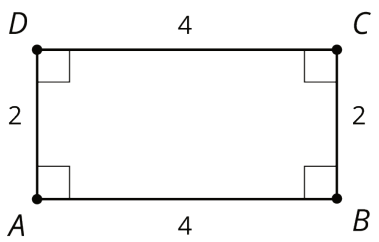


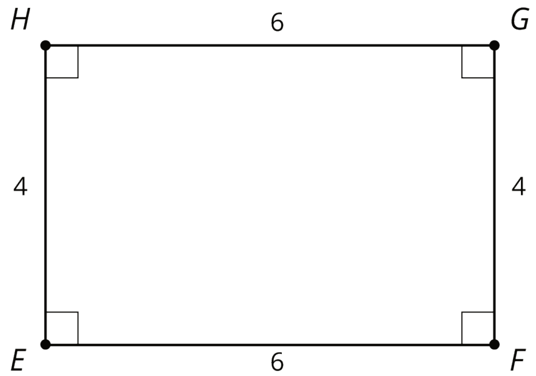


1. Let's look at a square and a rhombus.

   Priya says, "These polygons are similar because their side lengths are all the same." Clare says, "These polygons are not similar because the angles are different." Do you agree with either Priya or Clare? Explain your reasoning.

2. Now, let's look at rectangles $ABCD$ and $EFGH$.



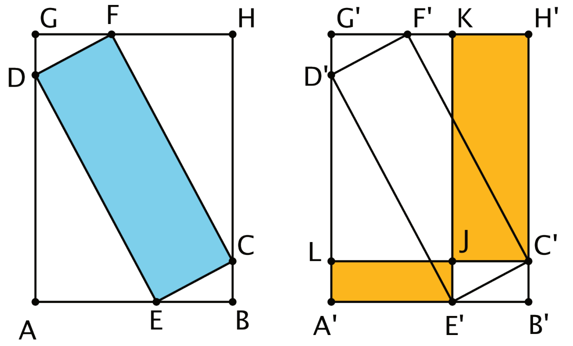


   Jada says, "These rectangles are similar because all of the side lengths differ by 2." Lin says, "These rectangles are similar. I can dilate AD and BC using a scale factor of 2 and AB and CD using a scale factor of 1.5 to make the rectangles congruent. Then I can use a translation to line up the rectangles." Do you agree with either Jada or Lin? Explain your reasoning.

Student Response

1. I agree with Clare. The angle measures in similar polygons are the same so these polygons can not be similar. Priya is not right: the side lengths of these polygons are the same but that is not enough to conclude that they are similar.
2. I disagree with both Jada and Lin. Jada is right that the side lengths all differ by 2 but the scale factor between the short sides of the rectangles and the scale factor between the long sides are not the same. This means that the two rectangles are not similar. Lin is not correct because the scale factor for dilating one set of sides can not be different than the scale factor for dilating the other set of sides.

Are You Ready for More?



Points A through H are translated to the right to create points A' through H'. All of the following are rectangles: $GHBA$, $FCED$, $KH'C'J$, and $LJE'A'$. Which is greater, the area of blue rectangle $DFCE$ or the total area of yellow rectangles $KH'C'J$ and $LJE'A'$?

Student Response

They are equal. You can show that rectangle $G'KJL$ is composed of triangles FHC and DEA, and rectangle $JC'B'E'$ is composed of triangles GFD and BEC.

Activity Synthesis

Invite selected students to share their choices and explanations, making sure that these points are highlighted:

1. If corresponding angles aren't congruent, then the figures cannot be similar (first problem).
2. If the side lengths are not all scaled by the same value, then the figures cannot be similar (second problem).

Unlike what Clare says, the same dilation (with the same scale factor) has to be performed to all parts of a figure to result in a similar figure. Another way to state the two important conclusions above is that for similar polygons:

- Corresponding angles are congruent.
- Corresponding side lengths are proportional.

Access for English Language Learners

Writing, Speaking, Listening: MLR1 Stronger and Clearer Each Time. After students have decided whether they agree with Jada or Lin, ask students to write a brief explanation of their reasoning. Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help them strengthen their ideas and clarify their language (e.g., "What does it mean for two polygons to be similar?", "Why is Jada's or Lin's reasoning incorrect?", or "How did you know the rectangles are not similar?", etc.). Students can borrow ideas and language from each partner to refine and clarify their original explanation. This will help students revise and refine their reasoning and their verbal and written output. *Design Principles(s): Optimize output (for explanation); Maximize meta-awareness*

7.3 Find Someone Similar

15 minutes

In the previous activity, students learn that in order to be similar, two figures must have congruent corresponding angles and proportional corresponding side lengths. In this activity, students apply this knowledge. Each student has a card with a figure on it and they identify someone with a similar (but not congruent) figure.

Monitor for students using these methods to identify a partner:

- Process of elimination (figures with different angle measures cannot be similar to one another)
- Looking for the same *kind* of figure (rectangle, square, rhombus)
- Looking for a the same kind of figure with congruent corresponding angles
- Looking for a the same kind of figure with corresponding sides scaled by the same scale factor

Addressing

- 8.G.A.4

Instructional Routines

- MLR2: Collect and Display

Launch

Distribute one card to each student. Explain that the task is for each student to find someone else in the class who has a similar (but not congruent) figure to their own, and be prepared to explain how they know the two figures are similar.

If the number of students in class is not a multiple of 10, ensure that any unused cards are matching pairs of similar figures. If there is an odd number of students, one or more students can be responsible for two cards, or some students can be appointed as referees.

Access for Students with Disabilities

Representation: Internalize Comprehension. Begin by providing students with generalizations to test for similarity. Make references to the conclusions from the previous activity and consider displaying examples and counterexamples of similar polygons, focusing on distinguishable properties of the figures.
Supports accessibility for: Conceptual processing

Access for English Language Learners

Conversing, Reading: MLR2 Collect and Display. As students figure out who has a card with a figure similar to their own, circulate and listen to students as they decide whether their figures are similar but not congruent. Write down the words and phrases students use to justify why the figures are or are not similar. Listen for students who state that similar figures must have congruent corresponding angles and corresponding sides scaled by the same scale factor. As students review the language collected in the visual display, encourage students to clarify the meaning of a word or phrase. For example, a phrase such as "the figures are not similar because the sides do not match" can be clarified by restating it as "the figures are not similar because the corresponding sides are not scaled by the same scale factor." This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-awareness of language.
Design Principle(s): Support sense-making; Maximize meta-awareness

Anticipated Misconceptions

If students have a hard time getting started, ask them to focus on properties of their figure that will be shared by a similar figure. For example, will a similar figure be a quadrilateral? Will a similar figure be square? A rectangle? A rhombus? What can you say about the angles in a similar figure?

Student Task Statement

Your teacher will give you a card. Find someone else in the room who has a card with a polygon that is similar but not congruent to yours. When you have found your partner, work with them to explain how you know that the two polygons are similar.

Student Response

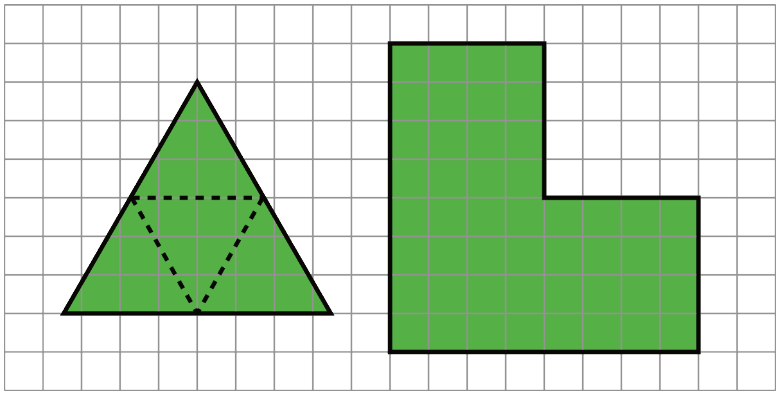
The similar quadrilaterals are:

- Rectangles with dimensions 6 by 8 and 9 by 12. Scale factor: $\frac{3}{2}$ or $\frac{2}{3}$

- Rectangles with dimensions 3 by 5 and 1.5 by 2.5. Scale factor: $\frac{1}{2}$ or 2
- Squares. Scale factor: $\frac{7}{8}$ or $\frac{8}{7}$
- Rhombuses with angles 60° and 120°. Scale factor: $\frac{9}{7}$ or $\frac{7}{9}$
- Rhombuses with angles 50° and 130°. Scale factor: $\frac{10}{9}$ or $\frac{9}{10}$

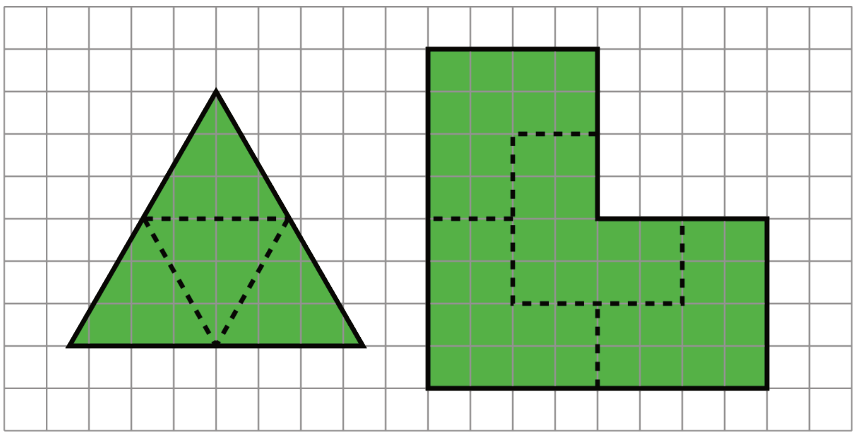
Are You Ready for More?

On the left is an equilateral triangle where dashed lines have been added, showing how you can partition an equilateral triangle into smaller similar triangles.



Find a way to do this for the figure on the right, partitioning it into smaller figures which are each similar to that original shape. What's the fewest number of pieces you can use? The most?

Student Response



The fewest you can use is 4 pieces, as in the image, each a $\frac{1}{2}$-scale dilation of the original. There is no upper limit on the number of pieces you could use. For example, you could take the four $\frac{1}{2}$-scale pieces and divide them each using four $\frac{1}{4}$-scale pieces in exactly the same pattern, then cover each of those with four $\frac{1}{8}$-scale pieces, etc.

Activity Synthesis

Invite selected students to share how they found a partner. Highlight different strategies: it is possible to proceed by process of elimination (this shape is a rectangle and mine is not so it is not a match) or by actively looking for certain features (e.g. a rhombus whose angle measures are 60 degrees and 120 degrees). Next ask how they knew their polygons were similar and which scale factor they used to show similarity.

Ask students what they looked for in searching for a partner with a similar figure. In particular ask

- "Were the side lengths important?" (For some figures, such as the rectangles, the side lengths were important.)
- "Were the angles important?" (Yes. Several of the figures had different angles and that meant they were not similar.)

Students may notice that *all* squares are similar since they have 4 right angles and proportional side lengths. On the other hand, this activity provides examples of rhombuses that are *not* similar to one another.

Lesson Synthesis

Important take aways from this lesson include:

- Similar figures have congruent corresponding angles *and* proportional corresponding side lengths.
- For some figures (like rectangles or squares), it is sufficient to focus on side lengths since corresponding angles are automatically congruent.
- For some figures (like rhombuses), it is sufficient to focus on angles since corresponding side lengths are automatically proportional.

Make sure students understand that, in general, determining whether two polygons are similar requires examining *both* side lengths *and* angles. On the other hand, it is possible to determine that two polygons are *not* similar by identifying a single pair of corresponding angles whose measures are different or a pair of corresponding side lengths with a different ratio than another pair of corresponding side lengths.

7.4 How Do You Know?

Cool Down: 5 minutes

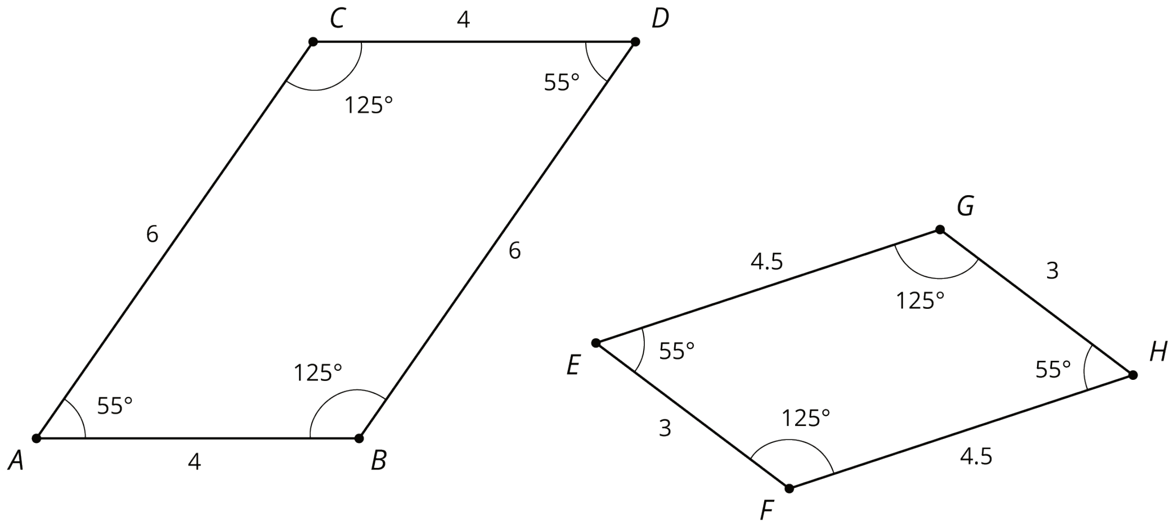
Students explain why two quadrilaterals are similar. Since they do not have criteria for when two parallelograms are similar, they will need to produce a sequence of rigid motions and dilations that take one figure to the other. They should be able to describe these motions rather than performing them.

Addressing

- 8.G.A.4

Student Task Statement

Explain how you know these two figures are similar.



Student Response

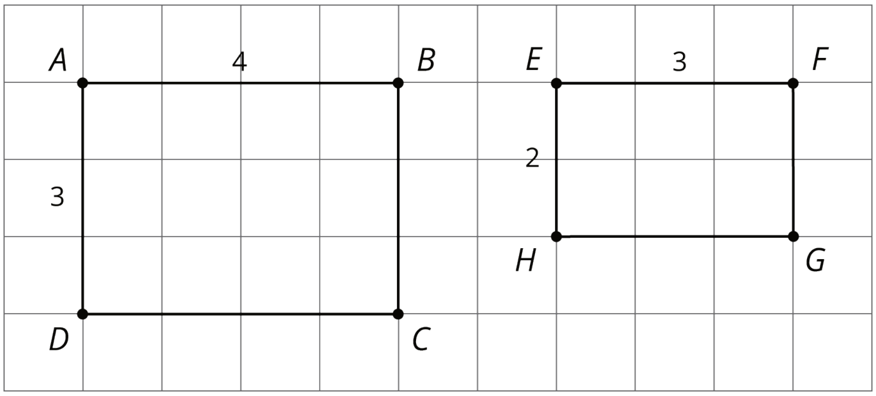
Dilating quadrilateral $ABCD$ with center A and scale factor $\frac{3}{4}$ gives a quadrilateral that is congruent to $EFGH$. This can be shown with a translation of A to E and then a rotation with center E.

Student Lesson Summary

When two polygons are similar:

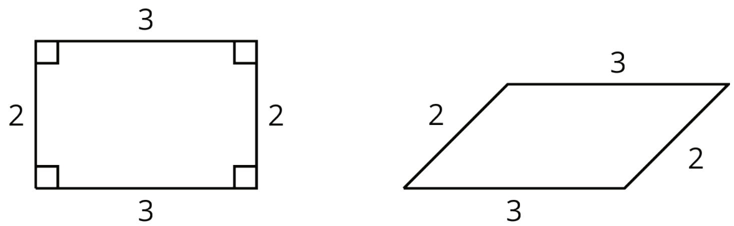
- Every angle and side in one polygon has a corresponding part in the other polygon.
- All pairs of corresponding angles have the same measure.
- Corresponding sides are related by a single scale factor. Each side length in one figure is multiplied by the scale factor to get the corresponding side length in the other figure.

Consider the two rectangles shown here. Are they similar?



It looks like rectangles $ABCD$ and $EFGH$ could be similar, if you match the long edges and match the short edges. All the corresponding angles are congruent because they are all right angles. Calculating the scale factor between the sides is where we see that "looks like" isn't enough to make them similar. To scale the long side AB to the long side EF, the scale factor must be $\frac{3}{4}$, because $4 \cdot \frac{3}{4} = 3$. But the scale factor to match AD to EH has to be $\frac{2}{3}$, because $3 \cdot \frac{2}{3} = 2$. So, the rectangles are not similar because the scale factors for all the parts must be the same.

Here is an example that shows how sides can correspond (with a scale factor of 1), but the quadrilaterals are not similar because the angles don't have the same measure:



Lesson 7 Practice Problems

Problem 1

Statement

Triangle DEF is a dilation of triangle ABC with scale factor 2. In triangle ABC, the largest angle measures 82°. What is the largest angle measure in triangle DEF?

A. 41°

B. 82°

C. 123°

D. 164°

Solution

B

Problem 2

Statement

Draw two polygons that are similar but could be mistaken for not being similar. Explain why they are similar.

Solution

Answers vary. Sample response: Two polygons with different orientations; two congruent polygons. Another sample response: Two polygons where a reflection is part of the transformation from one to the other.

Problem 3

Statement

Draw two polygons that are *not* similar but could be mistaken for being similar. Explain why they are not similar.

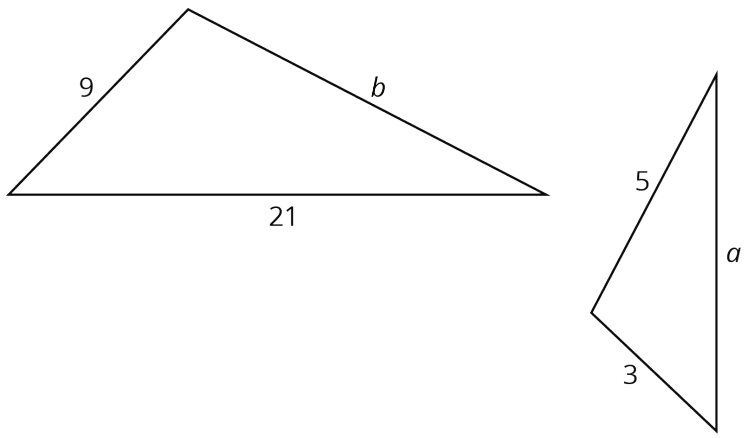
Solution

Answers vary. Sample response: Two polygons with the same angle measures but side lengths that are not proportional. Another sample response: Two polygons with proportional side lengths but incorrect angle measures.

Problem 4

Statement

These two triangles are similar. Find side lengths a and b. Note: the two figures are not drawn to scale.



Solution

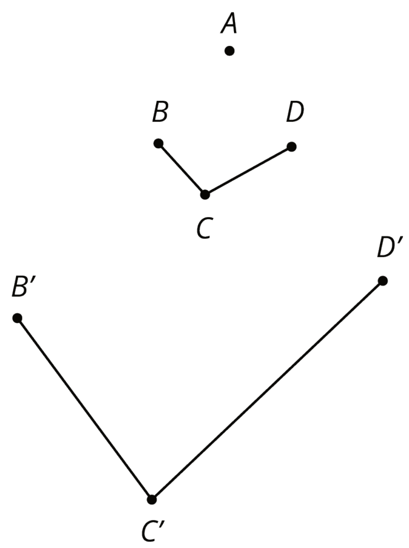
$a = 7, b = 15$

Problem 5

Statement

Jada claims that $B'C'D'$ is a dilation of BCD using A as the center of dilation.

What are some ways you can convince Jada that her claim is not true?



Solution

Answers vary. Below is a list of things which would have to be true if $B'C'D'$ is a dilation of BCD using A as the center of dilation. Any measurement which showed any of these to not hold is a complete answer.

- $m\angle BCD = m\angle B'C'D'$
- A, B, and B' should be collinear.
- A, C, and C' should be collinear.
- A, D, and D' should be collinear.
- $B'C'$ should be parallel to BC.
- $C'D'$ should be parallel to CD.

(From Unit 2, Lesson 3.)

Problem 6

Statement

a. Draw a horizontal line segment AB.

b. Rotate segment AB 90° counterclockwise around point A. Label any new points.

c. Rotate segment AB 90° clockwise around point B. Label any new points.

d. Describe a transformation on segment AB you could use to finish building a square.

Solution

a. Answers vary.

b. The segment is attached at point A and forms a right angle.

c. The segment is attached at point B and forms a right angle, parallel and in the same direction as the previous segment.

d. Answers vary. Sample response: Translate A to C.

(From Unit 1, Lesson 8.)