# Lesson 9: Side Length Quotients in Similar Triangles

## Goals

- Calculate unknown side lengths in similar triangles using the ratios of side lengths within the triangles and the scale factor between similar triangles.
- Generalize (orally) that the quotients of pairs of side lengths in similar triangles are equal.

## Learning Targets

- I can decide if two triangles are similar by looking at quotients of lengths of corresponding sides.
- I can find missing side lengths in a pair of similar triangles using quotients of side lengths.

## Lesson Narrative

In prior lessons, students learned that similar triangles are the images of each other under a sequence of rigid transformations and dilations, and that as a result, there is a scale factor that we can use to multiply all of the side lengths in one triangle to find the corresponding side lengths in a similar triangle. In this lesson, they will discover that if you determine the quotient of a pair of side lengths in one triangle, it will be equal to the quotient of the corresponding side lengths in a similar triangle. While this fact is not limited to triangles, this lesson focuses on the particular case of triangles so that students are ready to investigate the concept of slope in upcoming lessons.

### Alignments

### Building On

- 7.RP.A.2: Recognize and represent proportional relationships between quantities.

### Addressing

- 8.G.A: Understand congruence and similarity using physical models, transparencies, or geometry software.
- 8.G.A.4: Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

### Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR8: Discussion Supports

### Required Materials

**Geometry toolkits**
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

> **Student Learning Goals**
>
> Let's find missing side lengths in triangles.

# 9.1 Two-three-four and Four-five-six

**Warm Up: 5 minutes**
Two sets of triangle side lengths are given that do not form similar triangles. Students should recognize that there is no single scale factor that multiplies all of the side lengths in one triangle to get the side lengths in the other triangle.

### Addressing

- 8.G.A

### Launch

Give 2 minutes of quiet work time followed by a whole-class discussion.

### Anticipated Misconceptions

Students might think that adding the same number to each side length will result in similar triangles. Drawing a picture helps students see why this is not true.

> **Student Task Statement**
>
> Triangle $A$ has side lengths 2, 3, and 4. Triangle $B$ has side lengths 4, 5, and 6. Is Triangle $A$ similar to Triangle $B$?

### Student Response

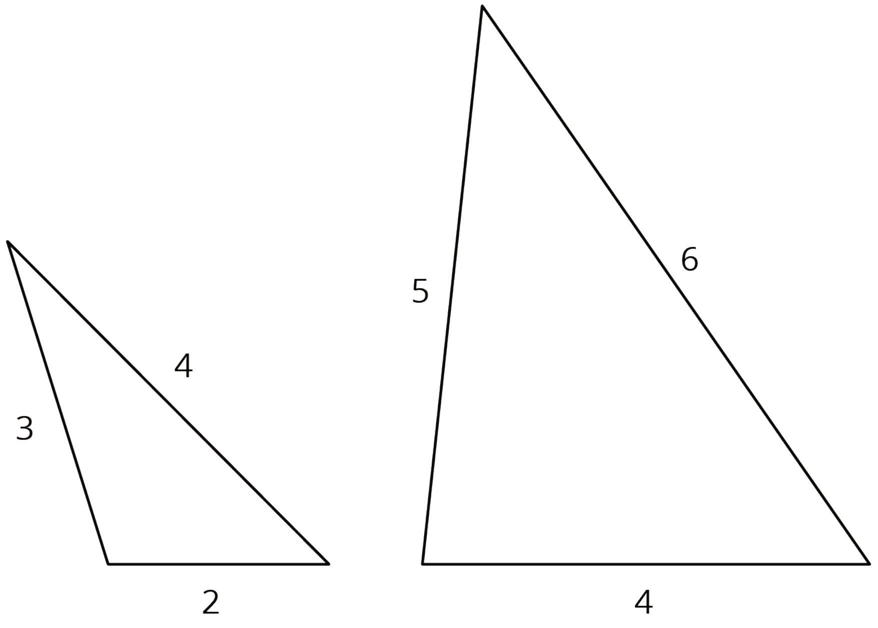
No. Sample explanations:

- If two figures are similar, then there is a single scale factor we can multiply all of the side lengths in one to get the side lengths in the other. Since doubling 2 gives 4, and doubling 3 gives 6, the third side in the second triangle would have to be 8 for the two to be similar.
- These triangles are not similar because we double the shortest side in Triangle A to get the shortest side in Triangle B, but we multiply the longest side in Triangle A by 1.5 to get the longest side in Triangle B. So the scale factor is not the same for all side lengths.

### Activity Synthesis

Ask students how they can tell without drawing a diagram. Make sure students understand that the triangles can not be similar because you can't apply the same scale factor to each side of one triangle to get the corresponding sides of the other triangle.

Display diagrams of the triangles for visual confirmation.




# 9.2 Quotients of Sides Within Similar Triangles

**15 minutes**

In previous lessons, students have seen that corresponding side lengths of similar polygons are proportional. That is, the side lengths in one polygon can be calculated by multiplying corresponding side lengths in a similar polygon by the same scale factor. This activity explores ratios of side lengths *within* triangles and how these compare for similar triangles. If $a$ and $b$ are the side lengths of a triangle then the corresponding side lengths of a similar triangle have lengths $sa$ and $sb$ for some positive scale factor $s$. This means that the ratios $a : b$ and $sa : sb$ are equivalent.

As students work, circulate to make sure that students have the correct values in the table, and address any misconceptions with individual groups as needed. Also watch for students who look to explain why the internal ratios of corresponding side lengths of similar triangles are equivalent and invite them to share their thinking during the discussion.

### Building On

- 7.RP.A.2

### Addressing

- 8.G.A

## Instructional Routines

- MLR8: Discussion Supports

## Launch

Arrange students in groups of 3. Assign one of the columns in the second table to one student in each group. Tell students, "Each group is going to compare side lengths in similar triangles. Work for 5 minutes by yourself. Then compare your findings with your partners."

### Access for Students with Disabilities

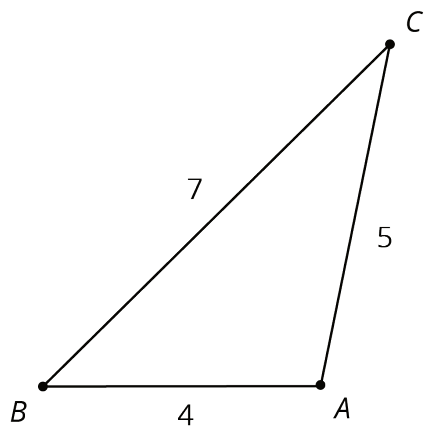
*Representation: Internalize Comprehension.* Activate or supply background knowledge. Allow students to use calculators to ensure inclusive participation in the activity.
*Supports accessibility for: Memory; Conceptual processing*

## Anticipated Misconceptions

If students find quotients in fraction form, they need to recognize that the fractions are equivalent.

### Student Task Statement

Triangle $ABC$ is similar to triangles $DEF$, $GHI$, and $JKL$. The scale factors for the dilations that show triangle $ABC$ is similar to each triangle are in the table.




1. Find the side lengths of triangles $DEF$, $GHI$, and $JKL$. Record them in the table.

| triangle | scale factor | length of short side | length of medium side | length of long side |
|---|---|---|---|---|
| $ABC$ | 1 | 4 | 5 | 7 |
| $DEF$ | 2 | | | |
| $GHI$ | 3 | | | |
| $JKL$ | $\frac{1}{2}$ | | | |

2. Your teacher will assign you one of the three columns. For all four triangles, find the quotient of the triangle side lengths assigned to you and record it in the table. What do you notice about the quotients?

| triangle | (long side) ÷ (short side) | (long side) ÷ (medium side) | (medium side) ÷ (short side) |
|---|---|---|---|
| $ABC$ | $\frac{7}{4}$ or 1.75 | | |
| $DEF$ | | | |
| $GHI$ | | | |
| $JKL$ | | | |

3. Compare your results with your partners' and complete your table.

**Student Response**

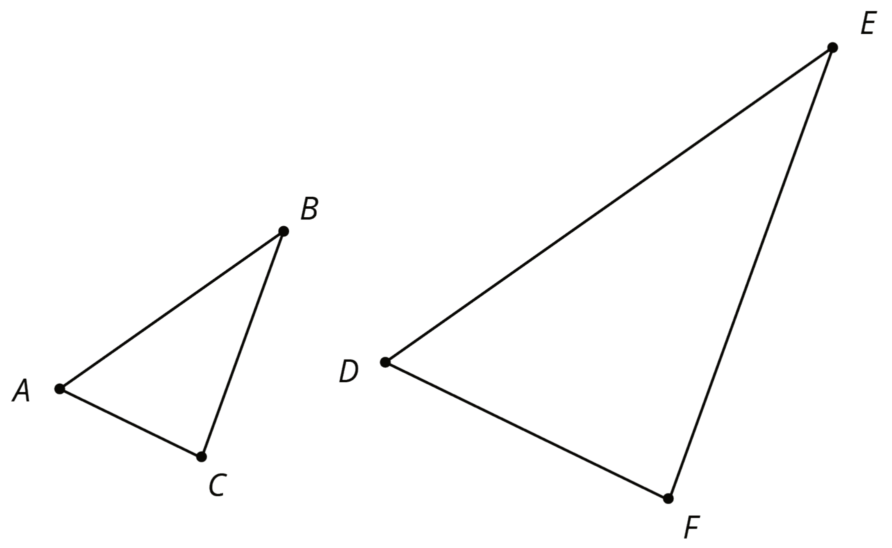
| triangle | scale factor | length of short side | length of medium side | length of long side |
|---|---|---|---|---|
| $ABC$ | 1 | 4 | 5 | 7 |
| $DEF$ | 2 | 8 | 10 | 14 |
| $GHI$ | 3 | 12 | 15 | 21 |
| $JKL$ | $\frac{1}{2}$ | 2 | 2.5 | 3.5 |

| triangle | (long side) ÷ (short side) | (long side) ÷ (medium side) | (medium side) ÷ (short side) |
|---|---|---|---|
| $ABC$ | $\frac{7}{4}$ or 1.75 | $\frac{7}{5}$ or 1.4 | $\frac{5}{4}$ or 1.25 |
| $DEF$ | $\frac{14}{8}$ or 1.75 | $\frac{14}{10}$ or 1.4 | $\frac{10}{8}$ or 1.25 |
| $GHI$ | $\frac{21}{12}$ or 1.75 | $\frac{21}{15}$ or 1.4 | $\frac{15}{12}$ or 1.25 |
| $JKL$ | $\frac{3.5}{2}$ or 1.75 | $\frac{3.5}{2.5}$ or 1.4 | $\frac{2.5}{2}$ or 1.25 |

The quotients in each column are the same.

### Are You Ready for More?

Triangles $ABC$ and $DEF$ are similar. Explain why $\frac{AB}{BC} = \frac{DE}{EF}$.




### Student Response

There is a scale factor $s$ such that $s \cdot AB = DE$ and $s \cdot BC = EF$. So $\frac{s \cdot AB}{s \cdot BC} = \frac{DE}{EF}$, and $\frac{AB}{BC} = \frac{DE}{EF}$.

### Activity Synthesis

The main takeaway from this activity is that quotients of corresponding side lengths in similar triangles are equal. Ask students for the triangles examined what the value of (medium side) ÷ (long side) would be? For the original triangle, it would be $\frac{5}{7}$, and students can check that this is the same value for the other triangles.

Ask students if they think the value of (medium side) ÷ (long side) would be $\frac{5}{7}$ for *any* triangle similar to $ABC$. Ask them to explain why. Help them to see that a triangle similar to $ABC$ will have side lengths $4s$, $5s$, and $7s$ for some (positive) scale factor $s$. The medium side divided by the long side will be $5s \div 7s = \frac{5s}{7s} = \frac{5}{7}$.

### Access for English Language Learners

*Speaking, Representing: MLR8 Discussion Supports.* At the end of the whole-class discussion, display this prompt for all to see, “The value of (medium side) ÷ (long side) will be ____ for any triangle similar to $ABC$ because…”. Give students 2–3 minutes to write a response. Invite students to read what they wrote to a partner as a way to rehearse what they will say when they share with the whole class. Rehearsing provides students with additional opportunities to speak and clarify their thinking. Listen for and amplify statements that use both formal and informal language such as, ratio, quotient, multiple, scale factor, reduce, simplify, and common factor. This will help students to explain their reasoning using appropriate language structure. *Design Principle(s): Optimize output (for explanation)*

# 9.3 Using Side Quotients to Find Side Lengths of Similar Triangles

**15 minutes**

In this activity, students calculate side lengths of similar triangles. They can use the scale factor between the similar triangles, studied in depth in previous lessons. Or they can look at internal ratios between corresponding side lengths within the triangles, introduced in the previous lesson. Students need to think strategically about which side lengths to calculate first since there are many missing values. As they discover more side lengths, this opens up more paths for finding the remaining values.

As students work, monitor for students who:

- Use scale factors between triangles.
- Notice that the long side is twice the short side in $GHI$ and use that to find $c$, $d$, or $e$.
- Notice that the long sides are equal in $ABC$ and use that to find $h$, $d$, or $e$.

Select students who use different strategies to find side lengths to share during the discussion.

## Addressing

- 8.G.A.4

## Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR8: Discussion Supports

## Launch

Tell students, “There are many ways to find the values of the unknown side lengths in similar triangles. Use what you have learned so far.” Give students 5 minutes of quiet work time followed by 5 minutes small group discussion and then a whole-class discussion.

### Access for Students with Disabilities

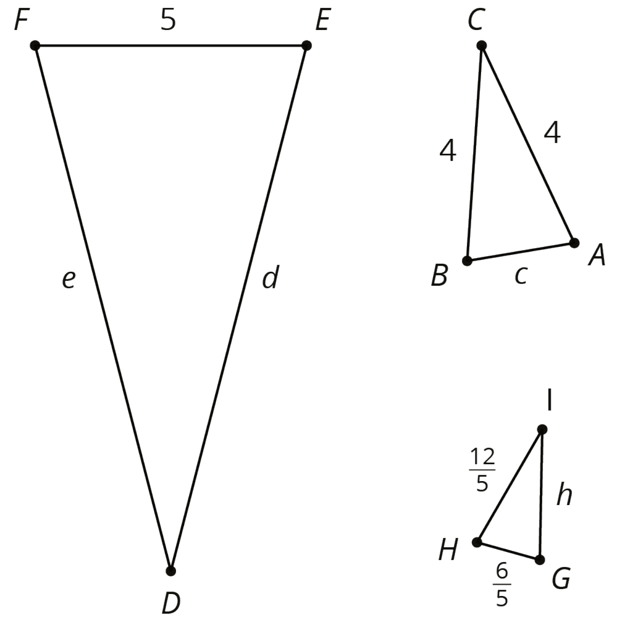
*Representation: Internalize Comprehension.* Demonstrate and encourage students to use color coding and annotations to highlight connections between representations in a problem. For example, use the same color for corresponding side lengths.
*Supports accessibility for: Visual-spatial processing*

## Anticipated Misconceptions

If students have trouble locating corresponding sides, suggest that they use tracing paper so they can rotate and or translate them. Another technique is to color corresponding side lengths the same color. For example, they could color $AB$, $EF$, and $GH$ all red.

## Student Task Statement

Triangles $ABC$, $EFD$, and $GHI$ are all similar. The side lengths of the triangles all have the same units. Find the unknown side lengths.




## Student Response

$c = 2$, $d = 10$, $e = 10$, $h = \frac{12}{5}$

### Activity Synthesis

Ask selected students to share the following strategies:

- using (external) scale factors to move from one triangle to another
- using quotients of corresponding side lengths within the triangles (internal scale factors)

Both methods are efficient and the method to use is guided by what information is missing and the numbers involved in the calculations. For example, if $h$ is the first missing value we find, then comparing with triangle $ABC$ and using internal scale factors is appropriate. To find $c$, again we can compare $ABC$ and $GHI$ and internal scale factors are appropriate again because $\frac{6}{5}$ is half of $\frac{12}{5}$ whereas comparing $\frac{12}{5}$ and 4 is more involved (the scale factor is $\frac{5}{3}$ from $\triangle GHI$ to $\triangle ABC$).

Ask students to articulate how they knew which sides of the similar triangles correspond. Make sure to make the following reasoning pathways explicit for all:

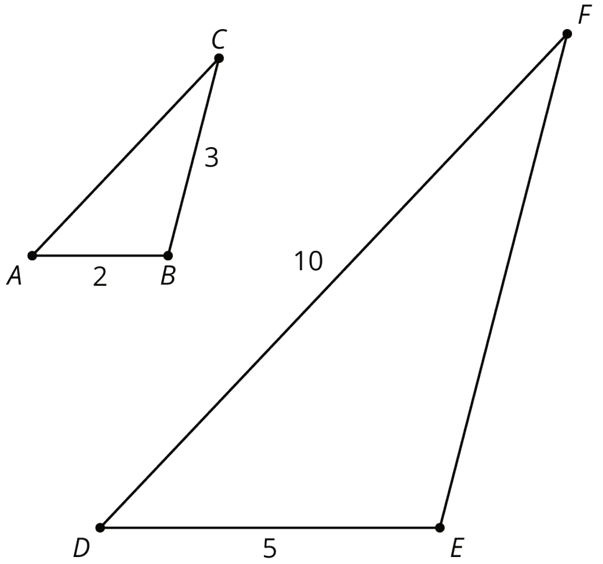
- Triangle $ABC$ has two equal side lengths, so the other two triangles will as well. This insight is efficient for finding $h$.
- One side of triangle $GHI$ is twice the length of another side, so this will be true of the other triangles as well. This insight is helpful for finding $c$, $d$, and $e$.

Emphasize that there are many different relationships that can be used to find side lengths of similar triangles.

#### Access for English Language Learners

*Speaking: MLR8 Discussion Supports:* To support students to articulate how they knew which sides of the similar triangles correspond, provide sentence frames such as, “For similar triangles _____ and _____ I know that sides _____ and _____ correspond because ________."
Sentence frames invite and incentivize more student participation, conversation, and meta-awareness of language. This routine should help students reason about the ratio of corresponding sides of similar triangles and communicate their understanding.
*Design Principle(s): Support sense-making, Optimize output (for comparison)*

## Lesson Synthesis




These two triangles are similar:Ask students what the scale factor is from $\triangle ABC$ to $\triangle DEF$. It's $\frac{5}{2}$ since sides $AB$ and $DE$ are corresponding sides. One way to find $AC$ would be to divide the length of $DF$ by the scale factor $\frac{5}{2}$ giving a length of 4. A simpler arithmetic way to do this is to notice that $DF$ is twice the length of $DE$. This means that $AC$ is twice the length of $AB$ (scaling $AC$ and $AB$ *both* by $\frac{5}{2}$ does not change their quotient!).

Sometimes both methods for calculating missing side lengths are equally effective. For $EF$, we can notice that it is $\frac{5}{2}$ the length of the corresponding side $BC$ so that's 7.5. Or we can notice that it is $\frac{3}{2}$ the length of $DE$, again 7.5 ($\frac{3}{2}$ is the quotient of the corresponding sides $BC$ and $AB$ in $\triangle ABC$).

# 9.4 Similar Sides

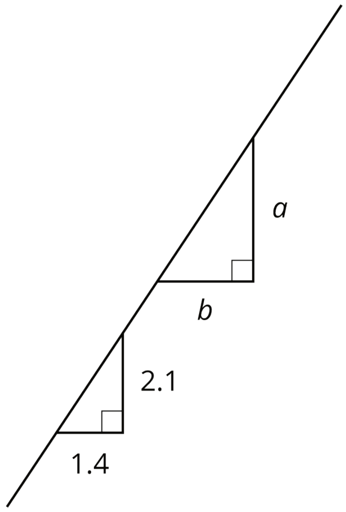
**Cool Down: 5 minutes**
Students apply the equality of internal ratios of sides on similar triangles to find missing side lengths. These particular triangles (slope triangles) will be a focus of study in the following lessons.

### Addressing

- 8.G.A

### Student Task Statement

The two triangles shown are similar. Find the value of $\frac{a}{b}$.



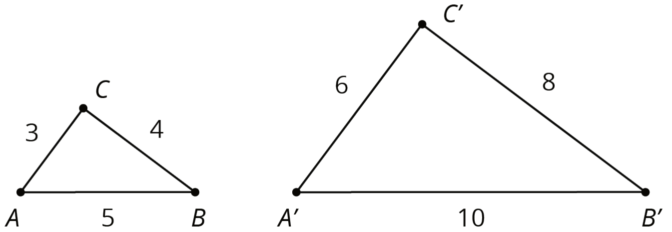

**Student Response**

$\frac{3}{2}$ or 1.5

## Student Lesson Summary

If two polygons are similar, then the side lengths in one polygon are multiplied by the same scale factor to give the corresponding side lengths in the other polygon.

For these triangles the scale factor is 2:



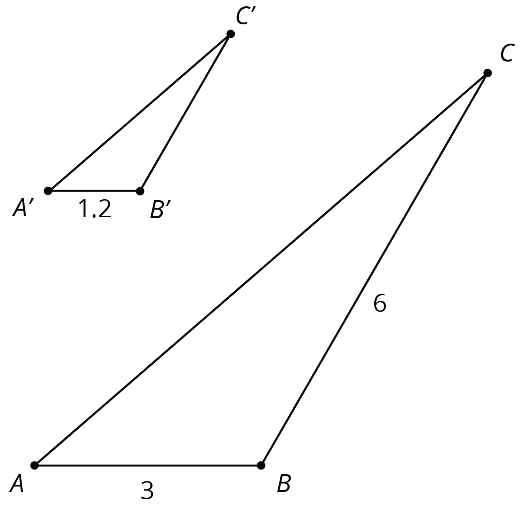

Here is a table that shows relationships between the short and medium length sides of the small and large triangle.

| | small triangle | large triangle |
|---|---|---|
| **medium side** | 4 | 8 |
| **short side** | 3 | 6 |
| **(medium side) ÷ (short side)** | $\frac{4}{3}$ | $\frac{8}{6} = \frac{4}{3}$ |

The lengths of the medium side and the short side are in a ratio of $4 : 3$. This means that the medium side in each triangle is $\frac{4}{3}$ as long as the short side. This is true for all similar polygons; the ratio between two sides in one polygon is the same as the ratio of the corresponding sides in a similar polygon.

We can use these facts to calculate missing lengths in similar polygons. For example, triangles $A'B'C'$ and $ABC$ shown here are similar. Let's find the length of segment $B'C'$.

In triangle $ABC$, side $BC$ is twice as long as side $AB$, so this must be true for any triangle that is similar to triangle $ABC$. Since $A'B'$ is 1.2 units long and $2 \cdot 1.2 = 2.4$, the length of side $B'C'$ is 2.4 units.

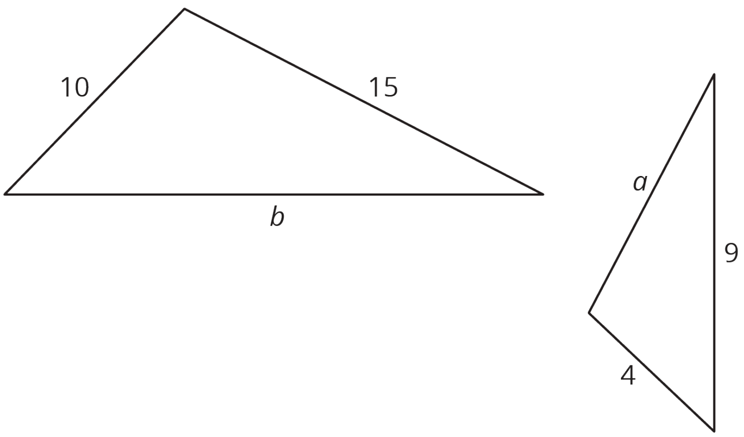



# Lesson 9 Practice Problems

## Problem 1

### Statement

These two triangles are similar. What are $a$ and $b$? Note: the two figures are not drawn to scale.



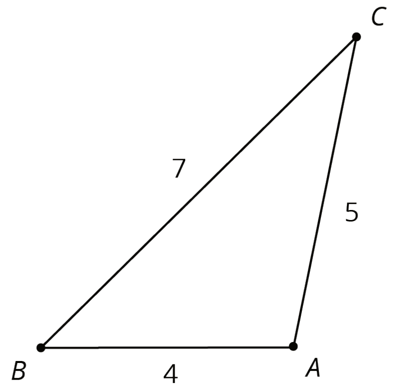

### Solution

$a = 6$, $b = 22.5$ (the scale factor between the triangles is 2.5)

## Problem 2

### Statement

Here is triangle $ABC$. Triangle $XYZ$ is similar to $ABC$ with scale factor $\frac{1}{4}$.




a. Draw what triangle $XYZ$ might look like.

b. How do the angle measures of triangle $XYZ$ compare to triangle $ABC$? Explain how you know.

c. What are the side lengths of triangle $XYZ$?

d. For triangle $XYZ$, calculate (long side) ÷ (medium side), and compare to triangle $ABC$.

## Solution

a. Answers vary.

b. The angle measures are the same, because in similar polygons, corresponding angles are congruent.

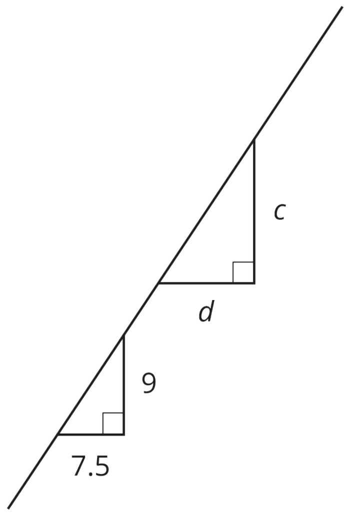
c. The side lengths are 1, $\frac{5}{4}$, and $\frac{7}{4}$.

d. The result is $\frac{7}{5}$, the same as the corresponding result for triangle $ABC$.

## Problem 3

### Statement

The two triangles shown are similar. Find the value of $\frac{d}{c}$.



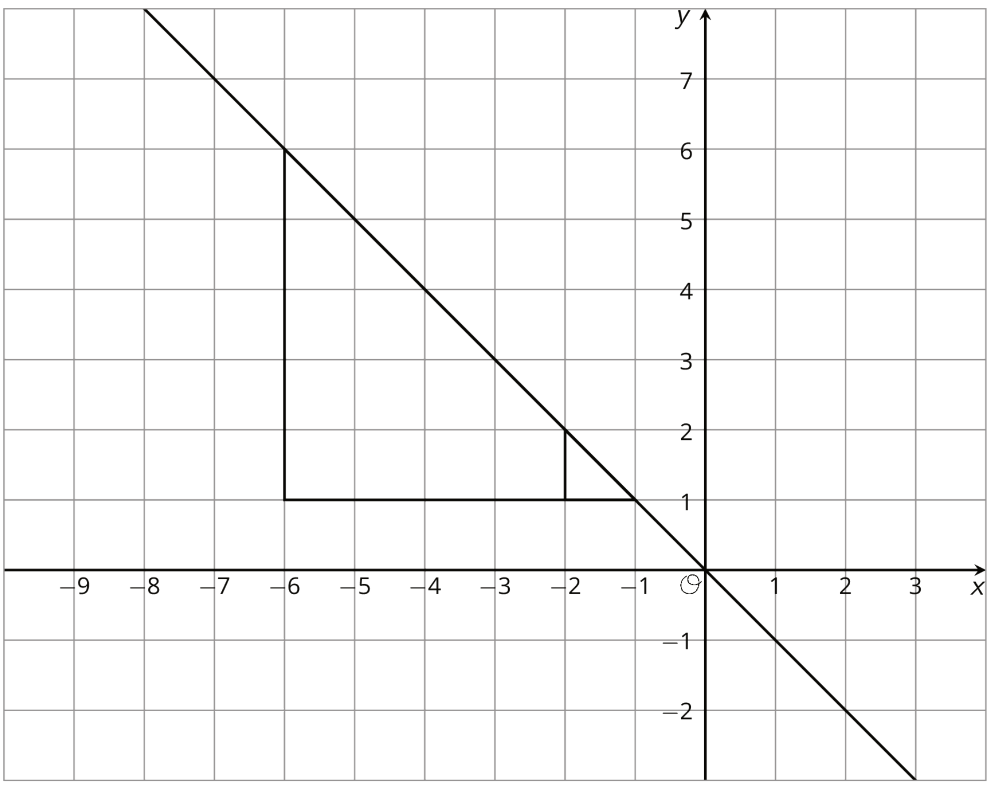

## Solution

$\frac{5}{6}$ (or equivalent)

## Problem 4

### Statement

The diagram shows two nested triangles that share a vertex. Find a center and a scale factor for a dilation that would move the larger triangle to the smaller triangle.



## Solution

Center: $(-1, 1)$, scale factor: $\frac{1}{5}$

(From Unit 2, Lesson 5.)