Lesson 1: Tiling the Plane

Goals

- Compare (orally) areas of the shapes that make up a geometric pattern.
- Comprehend that the word "area" (orally and in writing) refers to how much of the plane a shape covers.

Learning Targets

- I can explain the meaning of area.

Lesson Narrative

Students start the first lesson of the school year by recalling what they know about **area** (note that students studied the areas of rectangles with whole-number side lengths in grade 3 and with fractional side lengths in grade 5). The mathematics they explore is not complicated, so it offers a low threshold for entry. The lesson does, however, uncover two important ideas:

- If two figures can be placed one on top of the other so that they match up exactly, then they have the same area.
- The area of a **region** does not change when the region is decomposed and rearranged.

At the end of this lesson, students are asked to write their best definition of area. It is important to let them formulate their definition in their own words. For English learners, it is especially important that they be encouraged to use their own words and also to use words of their peers. In the next lesson, students will revisit the definition of area as the number of square units that cover a region without gaps or overlaps.

As the first set of problems in a problem-based curriculum, students will also begin their year-long work on making sense of problems and persevering in solving them (MP1). This opening lesson leaves space for teachers to begin setting classroom routines and their expectations for mathematical discourse (MP3).

In all of the lessons in this unit, students should have access to their geometry toolkits, which should contain tracing paper, graph paper, colored pencils, scissors, and an index card. Students may not need all (or even any) of these tools to solve a particular problem. However, to make strategic choices about when to use which tools (MP5), students need to have opportunities to make those choices. Apps and simulations should supplement rather than replace physical tools.

Notes on terminology. In these materials, when we talk about a figure such as a rectangle, triangle, or circle, we usually mean the boundary of the figure (e.g., the sides of the rectangle), not including the region inside. However, we also use shorthand language such as "the area of a rectangle" to mean the "the area of the region inside the rectangle." The term shape could refer to a figure with or without its interior. Although the terms figure, region, and shape are used without being defined precisely for students, help students understand that sometimes our focus is on the boundary (which in this unit will always be composed of black line segments), and sometimes it is on the region inside (which in this unit will always be shown in color and referred to as “the shaded region”).

Alignments

Building On

- 3.G.A: Reason with shapes and their attributes.

Building Towards

- 6.G.A.1: Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR2: Collect and Display
- Think Pair Share
- Which One Doesn’t Belong?

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Required Preparation

Assemble geometry toolkits. It would be best if students have access to these toolkits at all times throughout the unit. Toolkits include tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

> **Student Learning Goals**
>
> Let’s look at tiling patterns and think about area.

1.1 Which One Doesn’t Belong: Tilings

Warm Up: 10 minutes

This warm-up prompts students to compare four geometric patterns, explain their reasoning, and hold mathematical conversations. It allows you to hear how students use terminology in describing geometric characteristics.

Observing patterns gives every student an entry point. Each figure has at least one reason it does not belong. The patterns also urge students to think about shapes that cover the plane without gaps and overlaps, which supports future conversations about the meaning of area.

Before students begin, consider establishing a small, discreet hand signal that students can display to indicate they have an answer they can support with reasoning. This signal could be a thumbs-up, a certain number of fingers that tells the number of responses they have, or another subtle signal. This is a quick way to see if students have had enough time to think about the problem. It also keeps students from being distracted or rushed by hands being raised around the class.

Anticipate students to describe the patterns in terms of:

- Colors (blue, green, yellow, white, or no color)
- Size of shapes or other measurements
- Geometric shapes (polygons, squares, pentagons, hexagons, etc.)
- Relationships of shapes (whether each side of the polygons meets the side of another polygon, what polygon is attached to each side, whether there is a gap between polygons, etc.)

Building On

- 3.G.A

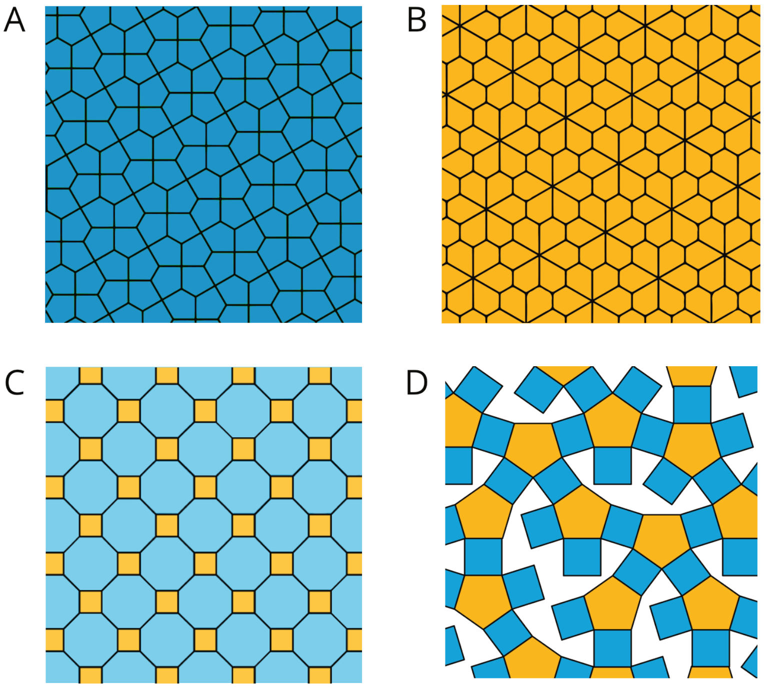
Instructional Routines

- Which One Doesn’t Belong?

Launch

Arrange students in groups of 2–4. Display the four patterns for all to see. Give students 1 minute of quiet think time and ask them to indicate when they have noticed one pattern that does not belong and can explain why. Encourage them to think of more than one possibility. When the minute is up, give students 2 minutes to share their response with their group, and then together find at least one reason, if possible, that each pattern doesn't belong.

> ### Student Task Statement
>
> Which pattern doesn’t belong?



Student Response

Answers vary. Sample responses:

- A: It doesn't have any yellow. Groups of four pentagons make hexagonal shapes that interlock without gaps.
- B: It doesn't have any blue. Groups of six pentagons make flower-like shapes that interlock without gaps.
- C: It doesn't have any pentagons. It has octagons and squares. The polygons that make up the patterns are very different in size.
- D: It has gaps between the shapes. Not all of the colored polygons meet another colored polygon on all sides. It has white (or non-filled) shapes that are more complex than other colored shapes. It is the only one where all the polygons have the same side length.

Activity Synthesis

After students shared their observations in groups, invite each group to share one reason why a particular figure might not belong. Record and display the responses for all to see. After each response, poll the rest of the class to see if others made the same observation.

Since there is no single correct answer to the question of which pattern does not belong, attend to students’ explanations, and make sure the reasons given are correct. Prompt students to explain the meaning of any terminology they use (names of polygons or angles, parts of polygons, area, etc.) and to substantiate their claims. For example, a student may claim that Pattern D does not belong because its polygons all have the same side length. Ask how they know that is the case, and whether that is true for the white (or non-filled) polygon.

Explain to students that covering a two-dimensional region with copies of the same shape or shapes such that there are no gaps or overlaps is called "tiling" the plane. Patterns A, B, and C are examples of tiling. Tell students that we explore more tilings in upcoming activities.

1.2 More Red, Green, or Blue?

25 minutes (there is a digital version of this activity)
This activity asks students to compare the amounts of the plane covered by two tiling patterns, with the aim of supporting two big ideas of the unit:

- If two figures can be placed one on top of the other so that they match up exactly, then they have the same area.
- A region can be decomposed and rearranged without changing its area.

Students are likely to notice that in each pattern:

- The same three polygons (triangles, rhombuses, and trapezoids) are used as tiles.
- The entire tiling pattern is composed of these hexagons.
- The shapes are arranged without gaps and overlaps, but their arrangements are different.
- A certain set of smaller tiles form a larger hexagon. Each hexagon has 3 trapezoids, 4 rhombuses, and 7 triangles.

Expect some students to begin their comparison by counting each shape, either within a hexagon or the entire pattern. To effectively compare how much of the plane is covered by each shape, however, they need to be aware of the relationships *between* the shapes. For example, two green triangles can be placed on top of a blue rhombus so that they match up exactly, which tells us that two green triangles cover the same amount of the plane as one blue rhombus. Monitor for such an awareness. (It is not necessary for students to use the word “area” in their explanations. At this point, phrases such as “they match up” or “two triangles make one rhombus” suffice.)

If students are not sure how to approach the questions, encourage them to think about whether any tools in their geometry toolkits could help. (For example, they could use tracing paper to trace entire patterns or certain shapes to make comparison, or use a straightedge to extend lines within the pattern. Some students may be inclined to cut out and compare the shapes.) Pattern tiles, if available, can be offered as well.

During the partner discussion, monitor for groups who discuss the following ideas so that they can share later, in this sequence:

- Relationships between two shapes: E.g., 2 triangles make a rhombus, and 3 triangles make a trapezoid.

- Relative overall quantities: E.g., there are 56 green triangles, 32 blue rhombuses (which have the same area as 64 triangles), and 24 red trapezoids (which have the same area as 72 triangles), so there is more red.
- Relative quantities in a hexagon: E.g., in each hexagon there are 7 green triangles, 4 rhombuses (which have the same area as 8 triangles), and 3 trapezoids (which have the same area as 9 triangles).

Classrooms using the digital activity have the option for students to use an applet that allows for the pattern to be isolated and also framed. This might assist students in focusing on how many of each shape comprise the pattern.

Building Towards

- 6.G.A.1

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR2: Collect and Display
- Think Pair Share

Launch

Arrange students in groups of 2. Ask one partner to analyze Pattern A and the other to analyze Pattern B. Tell students that their job is to compare the amount of the plane covered by each shape in the pattern.

Before students begin, introduce students to the geometry toolkits, and explain that they can use the toolkits for help, if needed. Give students 7–8 minutes of quiet think time. Then, ask students to share their responses with their partners and follow with a whole-class discussion.

Access for Students with Disabilities

Representation: Access for Perception. Provide physical objects such as pattern blocks for students to view or manipulate. Before students begin, introduce students to the geometry toolkits and pattern blocks, and ask students to brainstorm how they might use these tools to help them analyze each pattern. Monitor for students who make comparisons based on the number of shapes instead of accounting for the area covered by each shape. Use pattern blocks to clarify.
Supports accessibility for: Visual-spatial processing; Conceptual processing

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Access for English Language Learners

Conversing, Speaking, Listening: Math Language Routine 2 Collect and Display.
This is the first time Math Language Routine 2 is suggested as a support in this course. In this routine, the teacher circulates and listens to students talk while writing down the words, phrases, or drawings students produce. The language collected is displayed visually for the whole class to use throughout the lesson and unit. Generally, the display contains different examples of students using features of the disciplinary language functions, such as interpreting, justifying, or comparing. The purpose of this routine is to capture a variety of students' words and phrases in a display that students can refer to, build on, or make connections with during future discussions, and to increase students' awareness of language used in mathematics conversations.
Design Principle(s): Support sense-making

How It Happens:

1. After assigning students to work on Pattern A or B, circulate around the room and collect examples of language students are using to compare areas of polygons. Focus on capturing a variety of language describing the relationship between the size of two shapes, comparing overall quantities of shapes to equivalent areas of other shapes, and comparing relevant quantities in a hexagon. Aim to capture a range of student language that includes formal, precise, complete ideas and informal, ambiguous, and partial ideas. Plan to publicly update and revise this display throughout the lesson and unit.
   If pairs are stuck, consider using these questions to elicit conversation: "How many green triangles, blue rhombuses, and red trapezoids are in each pattern?", "Three triangles is equivalent to how many trapezoids?", and "Which shapes make up a hexagon?"

   If using the applet, have pairs use the applet together. Check that students focus on how many of each shape comprise the pattern by hiding, moving, and turning the shapes.

2. Create a display that includes visual representations of the words and phrases collected. Group language about Pattern A on one side of the display and language about Pattern B on the other side.

3. Close this conversation by posting the display in the front of the classroom for students to reference for the remainder of the lesson, and then have students move on to discussing other aspects of the activity. Continue to publicly update and revise the display throughout the lesson and unit.

Anticipated Misconceptions

Students may say more of the area is covered by the color they see the most in each image, saying, for example, "It just looks like there is more red." Ask these students if there is a way to prove their observations.

Students may only count the number of green triangles, red trapezoids, and blue rhombuses but not account for the area covered by each shape. If they suggest that the shape with the largest number of pieces covers the most amount of the plane, ask them to test their hypothesis. For example, ask, "Do 2 triangles cover more of the plane than 1 trapezoid?"

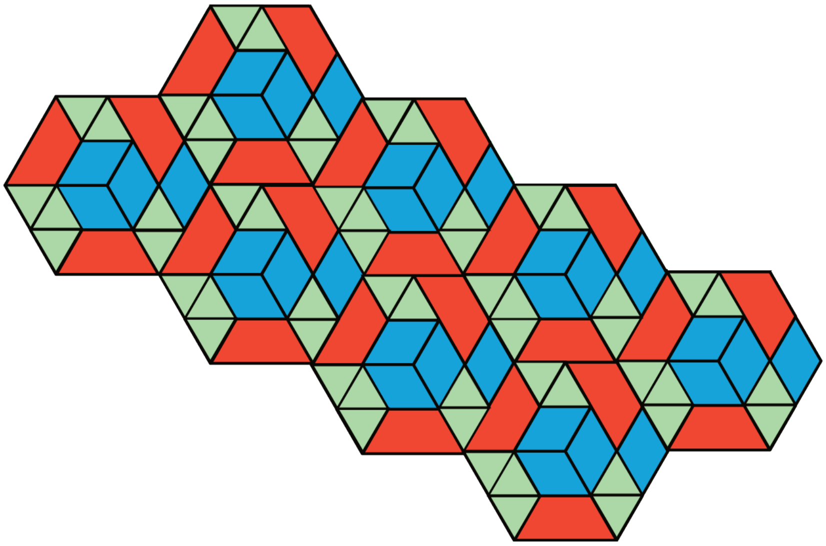
Students may not recall the terms trapezoid, rhombus, and triangle. Consider reviewing the terms, although they do not need to know the formal definitions to work on the task.

Student Task Statement

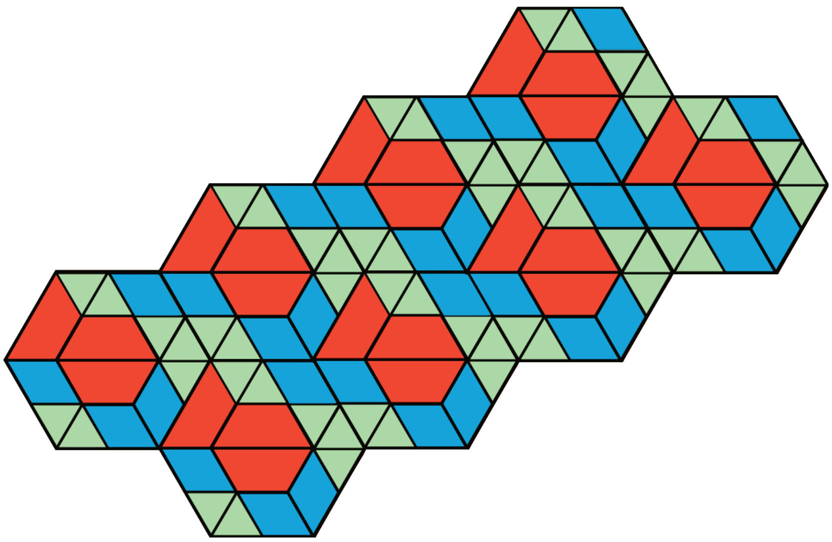
Your teacher will assign you to look at Pattern A or Pattern B.

In your pattern, which shape covers more of the plane: blue rhombuses, red trapezoids, or green triangles? Explain how you know.

Pattern A



Pattern B



Student Response

In both Patterns A and B, more of the plane is covered by red trapezoids than green triangles or blue rhombuses. Possible explanations:

- Patterns A and B are each made of 56 green triangles, 32 blue rhombuses, 24 red trapezoids.
    - One red trapezoid covers the same amount of the plane as 3 green triangles, so 24 red trapezoids cover the same amount of the plane as 72 green triangles, which are more than the 56 green triangles.
    - Each blue rhombus covers the same amount of the plane as 2 green triangles, so the 32 rhombuses cover the same amount of the plane as 64 green triangles, which are also more than the 56 green triangles.
- Each pattern is composed up of 8 hexagons. In each hexagon there are 3 red trapezoids, 4 blue rhombuses, and 7 green triangles.
    - Two red trapezoids can be arranged into a small hexagon. Three rhombuses can also be arranged into the same small hexagon. This means 2 trapezoids cover the same amount of the plane as 3 rhombuses.
    - Each large hexagon has 3 red trapezoids and 4 blue rhombuses. Since 2 trapezoids are equal to 3 rhombuses, we can just compare 1 trapezoid and 1 rhombus. We can see that 1 red trapezoid covers more of the plane than 1 rhombus.
    - Each large hexagon has 3 red trapezoids and 7 green triangles. One trapezoid covers the same amount of the plane as 3 triangles, so 3 trapezoids cover the same amount of the plane as 9 triangles, which are more than 7 green triangles.

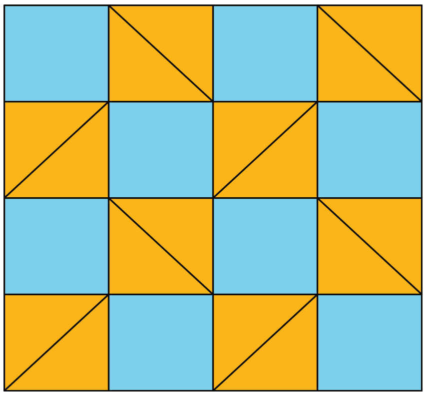
Are You Ready for More?

On graph paper, create a tiling pattern so that:

- The pattern has at least two different shapes.
- The same amount of the plane is covered by each type of shape.

Student Response

There are an infinite number of possibilities. Here is one:



Activity Synthesis

Select previously identified students or groups to share their answers and explanations. Sequence the explanations in the order listed in the Activity Narrative. To clarify the idea of comparing shapes by placing them on top of one another and seeing if or how they match, consider demonstrating using the digital applet.

Then, make it explicit that when we ask, “Which type of shape covers more of the plane?” we are asking them to compare the **areas** covered by the different types of shapes. To recast the comparisons of the shapes in terms of area, ask questions such as:

- “How does the area of the trapezoid compare to the area of the triangle?” (The area of the trapezoid is three times the area of the triangle.)
- “How does the area of the rhombus compare to the area of the triangle?” (The area of the rhombus is twice the area of the triangle.)
- “Is it possible to compare the area of the rhombuses in Pattern A and the area of the triangles in Pattern B? How?” (Yes, we can count the number of rhombuses in A and the number of triangles in B. Because 2 triangles have the same area as 1 rhombus, we divide the number of triangles by 2 to compare them.)

Lesson Synthesis

In this lesson, we have started to reason about what it means for two shapes to have the same **area**. We started doing mathematics and thinking about tools that can help us. Ask students:

- “What are some of the tools in the geometry toolkit and what are they used for?”
- “Draw two shapes that you know do not have the same area. How can you tell?”

Tell students that we will continue to think about area, to do and talk about mathematics, and to learn to use tools strategically.

1.3 What is Area?

Cool Down: 5 minutes

The purpose of this cool-down is to check how students are thinking about area after engaging in the activities. While the task prompts students to reflect on the work in this lesson, ideas about area from students' prior work in grades 3–5 may also emerge. Knowing the range of student thinking will help to inform the next day's lesson.

Student Task Statement

Think about your work today, and write your best definition of area.

Student Response

Answers vary. Sample responses:

- The amount of space inside a two-dimensional shape
- The measurement of the inside of a polygon
- The number of square units inside a shape

Student Lesson Summary

In this lesson, we learned about *tiling* the plane, which means covering a two-dimensional region with copies of the same shape or shapes such that there are no gaps or overlaps.

Then, we compared tiling patterns and the shapes in them. In thinking about which patterns and shapes cover more of the plane, we have started to reason about **area**.

We will continue this work, and to learn how to use mathematical tools strategically to help us do mathematics.

Glossary

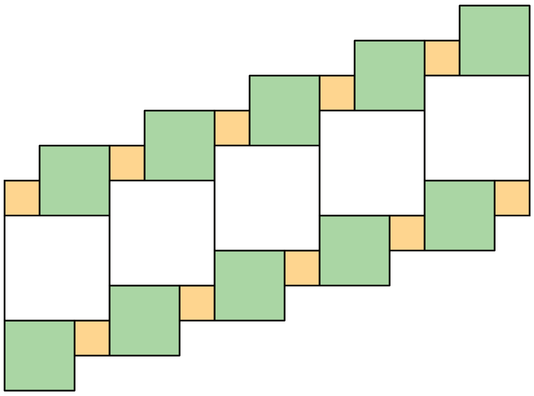
- area
- region

Lesson 1 Practice Problems

Problem 1

Statement

Which square—large, medium, or small—covers more of the plane? Explain your reasoning.



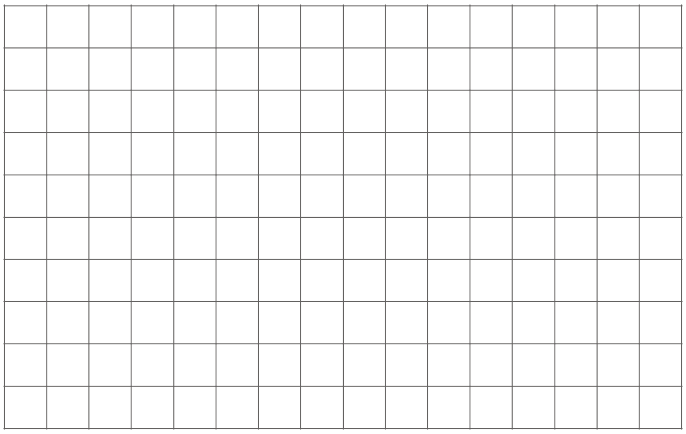
Solution

The large square covers more of the plane. Reasoning varies. Sample reasoning: A large square can fit exactly 9 small squares. A medium square can fit exactly 4 small squares. There are 5 large squares, which cover the same amount of the plane as 45 small squares. There are 10 medium squares, which cover the same amount of the plane as 40 small squares. There are only 10 small squares.

Problem 2

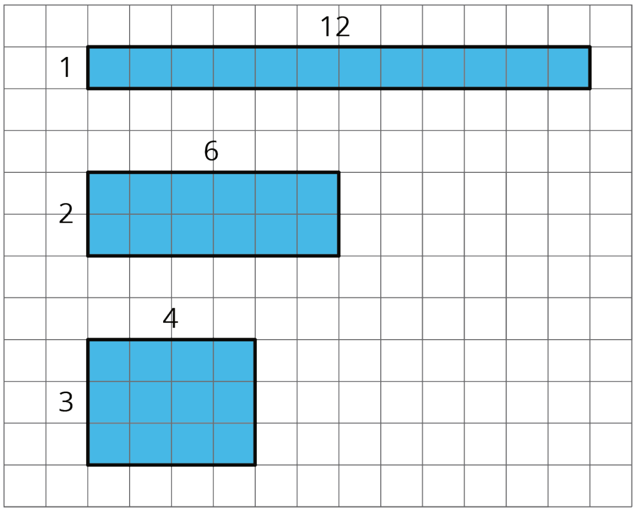
Statement

Draw three different quadrilaterals, each with an area of 12 square units.



Solution

Answers vary. Sample response:

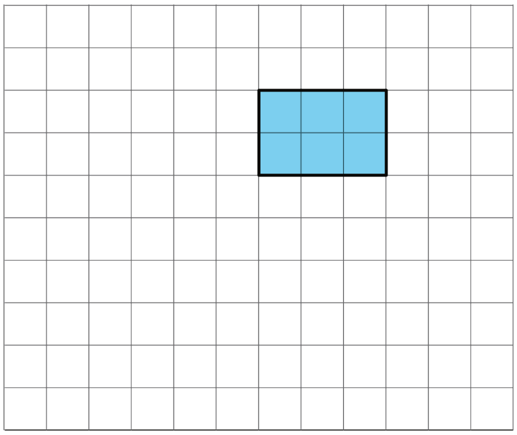


Problem 3

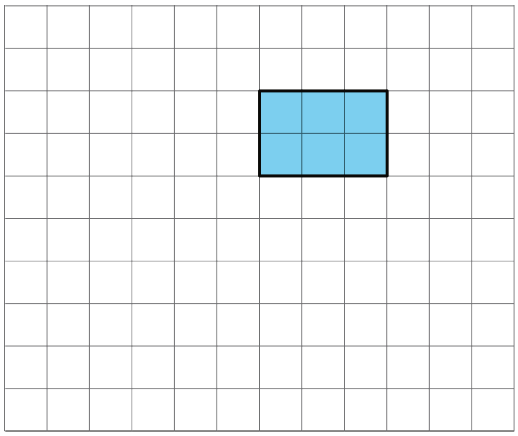
Statement

Use copies of the rectangle to show how a rectangle could:

a. tile the plane.

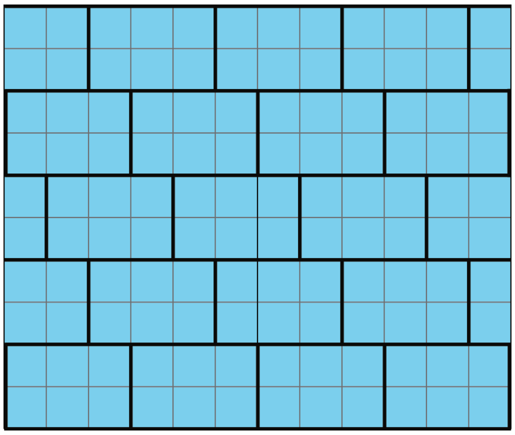


b. *not* tile the plane.

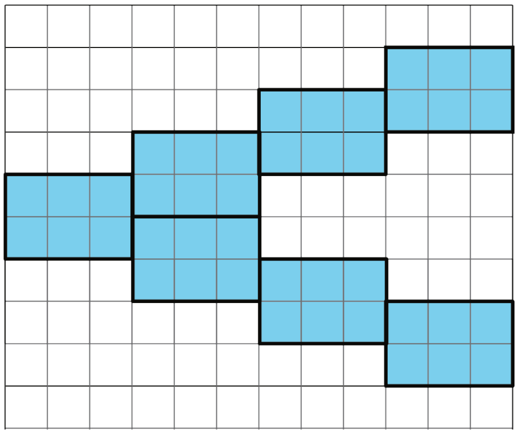


Solution

a. Answers vary. Sample response:



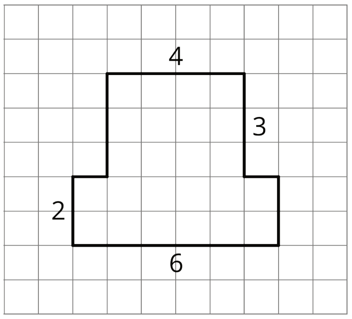
b. Answers vary. Sample response:



Problem 4

Statement

The area of this shape is 24 square units. Which of these statements is true about the area? Select **all** that apply.



A. The area can be found by counting the number of squares that touch the edge of the shape.

B. It takes 24 grid squares to cover the shape without gaps and overlaps.

C. The area can be found by multiplying the sides lengths that are 6 units and 4 units.

D. The area can be found by counting the grid squares inside the shape.

E. The area can be found by adding 4×3 and 6×2.

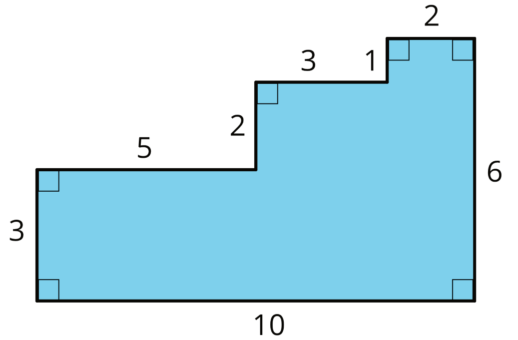
Solution

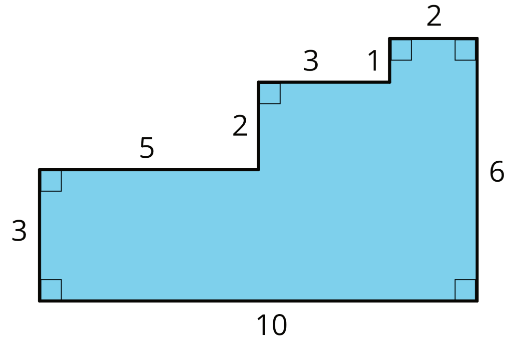
["B", "D", "E"]

Problem 5

Statement

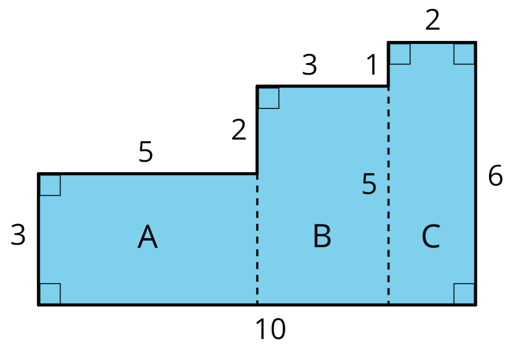
Here are two copies of the same figure. Show two different ways for finding the area of the shaded region. All angles are right angles.



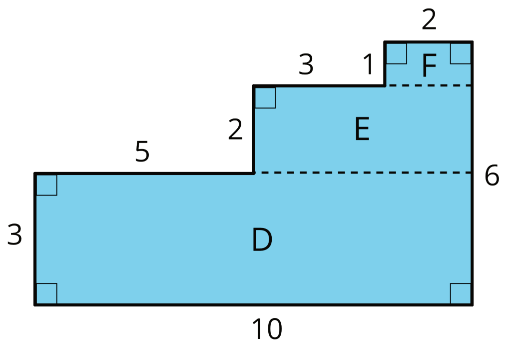


Solution

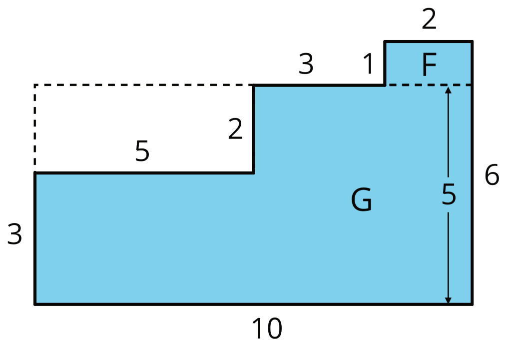
Answers vary. Sample strategies:



Area of A is 15 square units. Area of B is 15 square units. Area of C is 12 square units. The area of the entire region is $15 + 15 + 12$ or 42 square units.



Area of D is 30 square units. Area of E is 10 square units. Area of F is 2 square units. The area of the entire region is $30 + 10 + 2$ or 42 square units.



Area of F is 2 square units. Area of G is the area of the 10-by-5 rectangle subtracted by the area of a 5-by-2 rectangle in the upper left. $(10 \times 5) - (5 \times 2) = 50 - 10 = 40$, so the area of G is 40 square units. The total area is $40 + 2$ or 42 square units.

Problem 6

Statement

Which shape has a larger area: a rectangle that is 7 inches by $\frac{3}{4}$ inch, or a square with side length of $2\frac{1}{2}$ inches? Show your reasoning.

Solution

The square is larger. Its area is $2\frac{1}{2} \times 2\frac{1}{2} = \frac{5}{2} \times \frac{5}{2}$, which is $\frac{25}{4}$ or $6\frac{1}{4}$ square inches. The rectangle has an area of $5\frac{1}{4}$ square inches because $7 \times \frac{3}{4} = \frac{21}{4}$.