Lesson 4: Representing Functions at Rational Inputs

- Let's find how quantities are growing or decaying over fractional intervals of time.

4.1: Math Talk: Unknown Exponents

Solve each equation mentally.

1. $5^q = 125$

2. $\frac{1}{5^r} = \frac{1}{125}$

3. $5^t = \frac{1}{125}$

4. $125^u = 5$

4.2: Population of Nigeria

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In 1990, Nigeria had a population of about 95.3 million. By 2000, there were about 122.4 million people, an increase of about 28.4%. During that decade, the population can be reasonably modeled by an exponential function.

1. Express the population of Nigeria $f(d)$, in millions of people, d decades since 1990.

2. Write an expression to represent the population of Nigeria in 1996.

3. A student said, "The population of Nigeria grew at a rate of 2.84% every year."
    a. Explain or show why the student's statement is incorrect.

    b. Find the correct annual growth rate. Explain or show your reasoning.

4.3: Got Caffeine?

In healthy adults, caffeine has an average half-life of about 6 hours. Let's suppose a healthy man consumes a cup of coffee that contains 100 mg of caffeine at noon.

1. Each of the following expressions describes the amount of caffeine in the man's body some number of hours after consumption. How many hours after consumption?
   a. $100 \cdot \left(\frac{1}{2}\right)^1$
   b. $100 \cdot \left(\frac{1}{2}\right)^3$
   c. $100 \cdot \left(\frac{1}{2}\right)^{\frac{1}{6}}$
   d. $100 \cdot \left(\frac{1}{2}\right)^t$
2. a. Write a function g to represent the amount of caffeine left in the body, h hours after it enters the bloodstream.

   b. The function f represents the amount of caffeine left in the body after t 6-hour periods. Explain why $g(6) = f(1)$.

Are you ready for more?

What percentage of the initial amount of caffeine do you expect to break down in the first 3 hours: less than 25%, exactly 25%, or more than 25%? Explain or show your reasoning.

Lesson 4 Summary

Imagine a medicine has a half-life of 3 hours. If a patient takes 200 mg of the medicine, then the amount of medicine in their body, in mg, can be modeled by the function $f(t) = 200 \cdot \left(\frac{1}{2}\right)^t$. In this model, t represents a unit of time. Notice that the 200 represents the initial dose the patient took. The number $\frac{1}{2}$ indicates that for every 1 unit of time, the amount of medicine is cut in half. Because the half-life is 3 hours, this means that t must measure time in groups of 3 hours.

But what if we wanted to find the amount of medicine in the patient's body each hour after taking it? We know there are 3 equal groups of 1 hour in a 3-hour period. We also know that because the medicine decays exponentially, it decays by the same factor in each of those intervals. In other words, if b is the decay factor for each hour, then $b \cdot b \cdot b = \frac{1}{2}$ or $b^3 = \frac{1}{2}$. This means that over each hour, the medicine must decay by a factor of $\sqrt[3]{\frac{1}{2}}$, which can also be written as $\left(\frac{1}{2}\right)^{\frac{1}{3}}$. So if h is time in hours since the patient took the medicine, we can express the amount of medicine in mg, g, in the person's body as $g(h) = 200 \cdot \left(\sqrt[3]{\frac{1}{2}}\right)^h$ or $g(h) = 200 \cdot \left(\frac{1}{2}\right)^{\frac{h}{3}}$.