Lesson 2: Finding Area by Decomposing and Rearranging

Goals

- Calculate the area of a region by decomposing it and rearranging the pieces, and explain (orally and in writing) the solution method.
- Recognize and explain (orally) that if two figures can be placed one on top one other so that they match up exactly, they must have the same area.
- Show that area is additive by composing polygons with a given area.

Learning Targets

- I can explain how to find the area of a figure that is composed of other shapes.
- I know how to find the area of a figure by decomposing it and rearranging the parts.
- I know what it means for two figures to have the same area.

Lesson Narrative

This lesson begins by revisiting the definitions for **area** that students learned in earlier grades. The goal here is to refine their definitions (MP6) and come up with one that can be used by the class for the rest of the unit. They also learn to reason flexibly about two-dimensional figures to find their areas, and to communicate their reasoning clearly (MP3).

The area of two-dimensional figures can be determined in multiple ways. We can **compose** that figure using smaller pieces with known areas. We can **decompose** a figure into shapes whose areas we can determine and add the areas of those shapes. We can also decompose it and **rearrange** the pieces into a different but familiar shape so that its area can be found. The two key principles in this lesson are:

- Figures that match up exactly have equal areas. If two figures can be placed one on top of the other such that they match up exactly, then they have the same area.
- A figure can be decomposed and its pieces rearranged without changing its area. The sum of the areas of the pieces is equal to the area of the original figure. Likewise, if a figure is composed of non-overlapping pieces, its area is equal to the sum of the areas of the pieces. In other words, area is additive.

Students have used these principles since grade 3, but mainly to decompose squares, rectangles, and their composites (e.g., an L-shape) and rearrange them to form other such figures. In this lesson, they decompose triangles and rearrange them to form figures whose areas they know how to calculate.

A note about “two figures that match up exactly”: In grade 8, students will learn to refer to such figures as *congruent* and to describe congruence in terms of rigid motions (reflections, rotations, and translations). In these materials, the word congruent is not used in grade 6. A possibility is to use an informal term such as “identical,” so that students can talk about one figure being an “identical copy” of another. What “identical” means, however, might also require clarification (e.g., that it is independent of color and orientation).

Alignments

Building On

- 3.MD.C.5.b: A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.

Addressing

- 6.G.A.1: Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Building Towards

- 6.G.A: Solve real-world and mathematical problems involving area, surface area, and volume.

Instructional Routines

- MLR2: Collect and Display
- MLR8: Discussion Supports
- Think Pair Share

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Pre-assembled or commercially produced tangrams

Required Preparation

Prepare 1 set of tangrams that contains 4 small, 1 medium, and 2 large right triangles for every 2 students. Print and cut out the blackline master (printing on card stock is recommended), or use commercially-available tangrams. Note that the tangram pieces used here differs from a standard set in that two additional small triangles are used instead of a parallelogram.

A tangram applet is included for classrooms using the digital materials, but students can also be given the option of using physical tangrams instead of the digital tool.

Make sure students have access to their geometry toolkits, which should include tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

> **Student Learning Goals**
>
> Let’s create shapes and find their areas.

2.1 What is Area?

Warm Up: 10 minutes

This warm-up activates and refines students' prior knowledge of area. It prompts students to articulate a definitionof **area** that can be used for the rest of the unit. This definition of area is not new, but rather reiterates what students learned in grades 3–5.

Before this lesson, students explored tiling and tile patterns. Here, they analyze four ways a region is being tiled or otherwise fitted with squares. They decide which arrangements of squares can be used to find the area of the region and why, and use their analysis to write a definition of area. In identifying the most important aspects that should be included in the definition, students attend to precision (MP6).

Students' initial definitions may be incomplete. During partner discussions, note students who mention these components so they can share later:

- Plane or two-dimensional region
- Square units
- Covering a region completely without gaps or overlaps

Limit the whole-class discussion to 5–7 minutes to leave enough time for the work that follows.

Building On

- 3.MD.C.5.b

Building Towards

- 6.G.A

Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. Give students 2 minutes of quiet think time for the first question, and ask them to be ready to explain their decision. Then, give partners 3–4 minutes to share their responses and to complete the second question together.

Anticipated Misconceptions

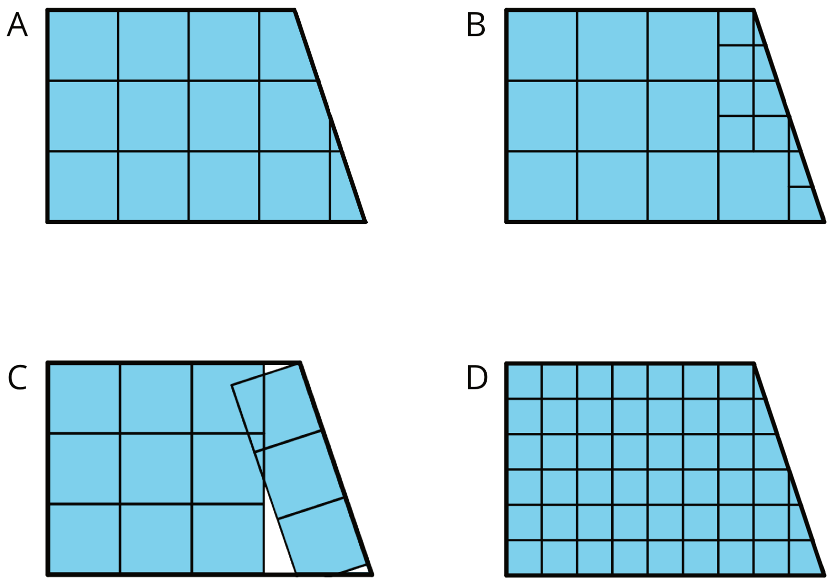
Students may focus on how they have typically found the area of a rectangle—by multiplying it side lengths—instead of thinking about what “the area of any region” means. Ask them to consider what the product of the side lengths of a rectangle actually tells us. (For example, if they say that the area of a 5-by-3 rectangle is 15, ask what the 15 means.)

Some students may think that none of the options, including A and D, could be used to find area because they involve partial squares, or because the partial squares do not appear to be familiar fractional parts. Use of benchmark fractions may help students see that the area of a region could be a non-whole number. For example, ask students if the area of a rectangle could be, say, $8\frac{1}{2}$ or $2\frac{1}{4}$ square units.

Student Task Statement

You may recall that the term **area** tells us something about the number of squares inside a two-dimensional shape.

1. Here are four drawings that each show squares inside a shape. Select **all** drawings whose squares could be used to find the area of the shape. Be prepared to explain your reasoning.



2. Write a definition of area that includes all the information that you think is important.

Student Response

1. A and D. B could be considered if the larger squares and the smaller ones are distinguished when determining area.
2. Answers vary, but the working definition should contain all of these components: “The area of a two-dimensional region (in square units) is the number of unit squares that cover the region without gaps or overlaps.”

Activity Synthesis

For each drawing in the first question, ask students to indicate whether they think the squares could or couldn't be used to find the area. From their work in earlier grades students are likely to see that the number of squares in A and D can each tell us about the area. Given the recent work on tiling, students may decide that C is unhelpful. Discuss students' decisions and ask:

- “What is it about A and D that can help us find the area?” (The squares are all the same size. They are unit squares.)
- “What is it about C that might make it unhelpful for finding area?” (The squares overlap and do not cover the entire region, so counting the squares won't give us the area.)
- “If you think B cannot be used to find area, why not?” (We can't just count the number of squares and say that the number is the area because the squares are not all the same size.)
- “If you think we can use B to find area, how?” (Four small squares make a large square. If we count the number of large squares and the number of small squares separately, we can convert one to the other and find the area in terms of either one of them.)

If time permits, discuss:

- “How are A and D different?” (A uses larger unit squares and D uses smaller ones. Each size represent a different unit.)
- “Will they give us different areas?” (They will give us areas in different units, such as square inches and square centimeters.)

Select a few groups to share their definitions of area or what they think should be included in the class definition of area. The discussion should lead to a definition that conveys key aspects of area: The area of a two-dimensional region (in square units) is the number of unit squares that cover the region without gaps or overlaps.

Display the class definition and revisit as needed throughout this unit. Tell students this will be a working definition that can be revised as they continue their work in the unit.

2.2 Composing Shapes

25 minutes (there is a digital version of this activity)

In grade 3, students recognized that area is additive. They learned to find the area of a rectilinear figure by decomposing it into non-overlapping rectangles and adding their areas. Here students extend that understanding to non-rectangular shapes. They compose tangram pieces—consisting of triangles and a square—into shapes with certain areas. The square serves as a unit square. Because students have only one square, they need to use these principles in their reasoning:

- If two figures can be placed one on top of the other so that they match up exactly, then they have the same area.
- If a figure is decomposed and rearranged to compose another figure, then its area is the same as the area of the original figure.

Each question in the task aims to elicit discussions about these two principles. Though they may seem obvious, these principles still need to be stated explicitly (at the end of the lesson), as more-advanced understanding of the area of complex figures depends on them.

The terms **compose**, **decompose**, and **rearrange** will be formalized in an upcoming lesson, but throughout this lesson, look for opportunities to demonstrate their use as students describe their work with the tangram pieces. When students use “make” or “build,” “break,” and “move around,” recast their everyday terms using the more formal ones.

As students work, notice how they compose the pieces to create shapes with certain areas. Look for students whose reasoning illustrates the ideas outlined in the Activity Synthesis.

Demonstrate the use of the word “compose” by repeating students' everyday language use and then recast using the formal terms here.

Addressing

- 6.G.A.1

Instructional Routines

- MLR2: Collect and Display
- Think Pair Share

Launch

Give each group of 2 students the following set of tangram pieces from the blackline master or from commercially available sets. Note that the tangram pieces used here differ from a standard set in that two additional small triangles are used instead of a parallelogram.

- Square: 1
- Small triangles: 4
- Medium triangle: 1
- Large triangles: 2

It is important not to give them more than these pieces.

Give students 2–3 minutes of quiet think time for the first three questions. Ask them to pause afterwards and compare their solutions to their partner's. If they created the same shape for each question, ask them to create a different shape that has the same given area before moving on. Then, ask them to work together to answer the remaining questions.

Classrooms using the digital activities can use physical tangram pieces or an applet with the same shapes to determine the relationships between the areas. Applet is adapted from the work of Harry Drew in GeoGebra.

Access for English Language Learners

Speaking, Conversing: MLR2 Collect and Display. Circulate and listen to the ways students describe composing, decomposing, and rearranging the shapes. On a display, write down common phrases you hear students say about each, such as "building," "breaking apart," "moving." Include relevant pictures or drawings. Update the display as needed throughout the remainder of the lesson. Remind students to borrow language from the display as needed. This will help students use mathematical language during their paired and whole-group discussions. *Design Principle(s): Maximize meta-awareness; Support sense-making*

Anticipated Misconceptions

Students may consider the area to be the number of pieces in the compositions, instead of the number of square units. Remind them of the meaning of area, or prompt them to review the definition of area discussed in the warm-up activity.

Because the 2 large triangles in the tangram set can be arranged to form a square, students may consider that square to be the square unit rather than the smaller square composed of 2 small triangles. Ask students to review the task statement and verify the size of the unit square.

Student Task Statement

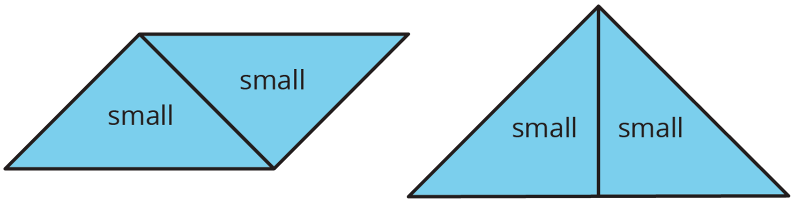
Your teacher will give you one square and some small, medium, and large right triangles. The area of the square is 1 square unit.

1. Notice that you can put together two small triangles to make a square. What is the area of the square composed of two small triangles? Be prepared to explain your reasoning.
2. Use your shapes to create a new shape with an area of 1 square unit that is not a square. Trace your shape.
3. Use your shapes to create a new shape with an area of 2 square units. Trace your shape.
4. Use your shapes to create a *different* shape with an area of 2 square units. Trace your shape.

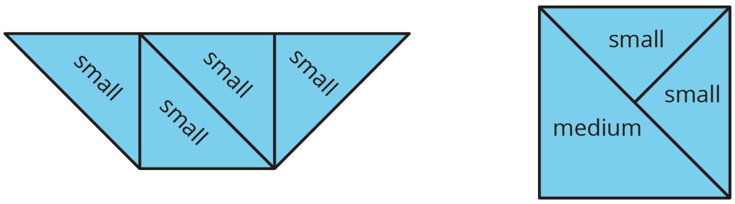
5. Use your shapes to create a new shape with an area of 4 square units. Trace your shape.

Student Response

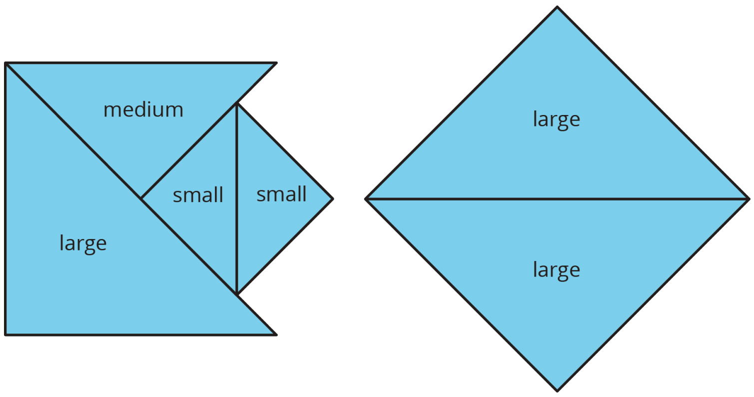
1. The area of the square made from two small triangles is 1 square unit because it is identical to the given square with area 1 square unit. “Identical” means you can put one on top of the other and they match up exactly.
2. Any composite of two small triangles.



3. Any composite of four small triangles or two small triangles and one medium triangle. Sample responses:



4. Any composite of four small triangles or two small triangles and one medium triangle.
5. Any composite with an area of 4 square units. Some possibilities:

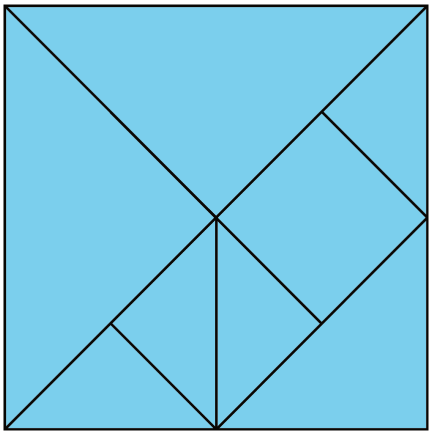


Are You Ready for More?

Find a way to use all of your pieces to compose a single large square. What is the area of this large square?

Student Response

The area is 8 square units. Sample response:



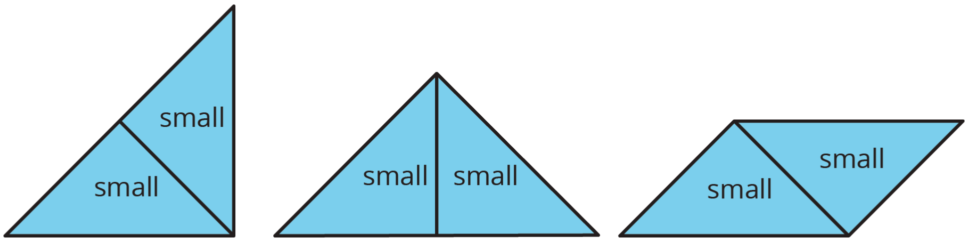
Activity Synthesis

Invite previously identified students (whose work illustrates the ideas shown here) to share. Name these moves explicitly as they come up: **compose**, **decompose**, and **rearrange**.

- First question: Two small triangles can be composed into a square that matches up exactly with the given square piece. This means that the two squares—the composite and the unit square—have the same area.

Tell students, “We say that if a region can be placed on top of another region so that they match up exactly, then they have the same **area**.”

- Second question: Two small triangles can be rearranged to compose a different figure but the area of that composite is still 1 square unit. These three shapes—each composed of two triangles—have the same area. If we rotate the first figure, it can be placed on top of the second so that they match up exactly. The third one has a different shape than the other two, but because it is made up of the same two triangles, it has the same area.



Emphasize: “If a figure is decomposed and rearranged as a new figure, the **area** of the new figure is the same as the area of the original figure.”

- Third and fourth questions: The composite figures could be formed in several ways: with only small triangles, with two triangles and a medium triangle, or with two small triangles and a square.

- Last question: A large triangle is needed here. To find its area, we need to either compose 4 smaller triangles into a large triangle, or to see that the large triangle could be decomposed into 4 smaller triangles, which can then be composed into 2 unit squares.

Access for Students with Disabilities

Representation: Develop Language and Symbols. Create a display of important terms and vocabulary. Invite students to suggest language or diagrams to include that will support their understanding of these terms. Include the following terms and maintain the display for reference throughout the unit: area, compose, decompose, and rearrange.
Supports accessibility for: Conceptual processing; Language

2.3 Tangram Triangles

Optional: 15 minutes (there is a digital version of this activity)
In this activity, students use the areas of composite shapes from the previous activity to reason about the area of each tangram shape. Students may have recognized previously that the area of one small triangle is $\frac{1}{2}$ square unit, the area of one medium triangle is 1 square unit, and the area of one large triangle is 2 square units. Here they practice articulating how they know that these observations are true (MP3). The explanations could be written in words, or as clearly-labeled illustrations that support their answers.

As partners discuss, look for two ways of thinking about the area of each assigned triangle: by *composing* copies of the triangle into a square or a larger triangle, or by *decomposing* the triangle or the unit square into smaller pieces and *rearranging* the pieces. Identify at least one student who uses each approach.

Addressing

- 6.G.A.1

Instructional Routines

- MLR8: Discussion Supports
- Think Pair Share

Launch

Arrange students in groups of 2. Assign the first and second questions to one partner and the second and third questions to the other partner. Give each group access to the geometry toolkits and the same set of tangram pieces as used in the earlier activity.

Give students 3–4 minutes of quiet time to find the areas of their assigned triangles and to construct their explanations, followed by a few minutes to share their responses with their partner. Tell students that as one partner explains, the other should listen carefully and either agree or disagree with the explanation. They should then come to an agreement about the answers and explanations.

Classrooms using the digital activities can use an applet to assist in determining the areas of the triangles.

Access for English Language Learners

Speaking, Writing: MLR8 Discussion Supports. Use this routine when students compare areas of triangles and squares to support the use of mathematical language. As students share their responses with their partner, circulate and encourage listeners to push speakers to use the language “compose,” “decompose,” or “rearrange” in their explanations. Look for students who name the square or larger triangles “composite figures” and amplify this language. Encourage students to borrow words and phrases from each other and to use this language in their written responses.
Design Principle(s): Cultivate conversation; Optimize output (for explanation)

Anticipated Misconceptions

If students initially have trouble determining the areas of the shapes, ask how they reasoned about areas in the previous activity. Have samples of composed and decomposed shapes that form one square unit available for students to reference.

Student Task Statement

Recall that the area of the square you saw earlier is 1 square unit. Complete each statement and explain your reasoning.

1. The area of the small triangle is ________ square units. I know this because . . .
2. The area of the medium triangle is ________ square units. I know this because . . .
3. The area of the large triangle is ________ square units. I know this because . . .

Student Response

1. $\frac{1}{2}$ square unit. Sample explanations:
    - Two small triangles can be put together to make a square, which has an area of one square unit. Because this composite shape matches the unit square exactly, their areas must be equal. This means that the area of each small triangle is half the area of the unit square.
    - A square can be decomposed into exactly two small triangles, so the area of each small triangle must be half of that of the square.
2. 1 square unit. Sample explanations:
    - Two small triangles can be put together to make one medium triangle. Two triangles can also be put together to make a square with an area of 1 square unit. Because two small

triangles make both a medium triangle and a square, the area of the medium triangle must be 1 square unit.

  - One medium triangle can be decomposed into two small triangles. These can be rearranged into a square whose area is 1 square unit, so the area of the medium triangle is also 1 square unit.

3. 2 square units. Sample explanations:

  - Two medium triangles can be arranged into one large triangle. Because the area of the medium triangle is 1 square unit, a figure that is composed of two of them has area 2 square units.
  - A large triangle can be decomposed into 4 small triangles, which can in turn be rearranged into two squares. The combined area of the two squares is 2 square units.

Activity Synthesis

After partners shared and agreed on the correct areas and explanations, discuss with the class:

- “Did you and your partner use the same strategy to find the area of each triangle?”
- “How were your explanations similar? How were they different?”

Select two previously identified students to share their explanations: one who reasoned in terms of *composing* copies of their assigned triangle into another shape, and one who reasoned in terms of *decomposing* their triangle or the unit square into smaller pieces and *rearranging* them. If these approaches are not brought up by students, be sure to make them explicit at the end of the lesson.

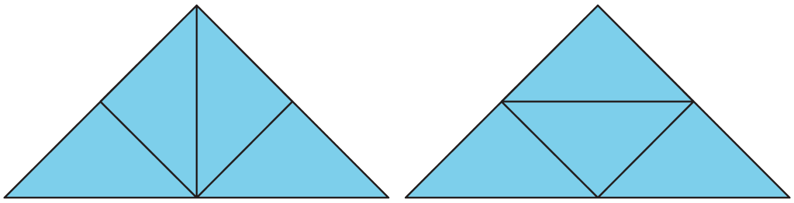
Lesson Synthesis

There are two principles that can help us reason about area:

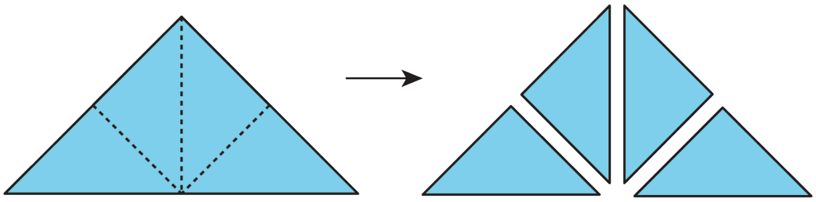
1. If two figures can be placed one on top of the other so that they match up exactly, then they have the same area.
2. The area of a figure can be found by adding the areas of its parts. If we **compose** (put together) a new figure from smaller pieces without overlapping them, then the sum of the areas of the pieces is the area of the new figure. Likewise, if we **decompose** (cut or break apart) a given figure into pieces, then the area of the given figure is the sum of the areas of the pieces. Even if we **rearrange** the pieces, the overall area does not change.

Here is an example. Suppose we know the area of a small triangle and wish to find the area of a large triangle. Demonstrate the following (using the tangram pieces, if possible):

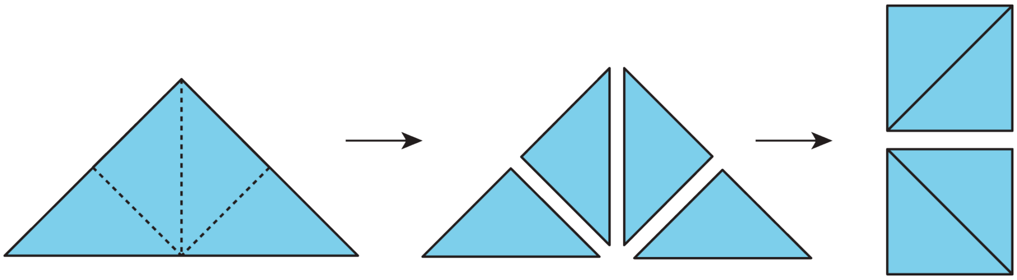
- We can use 4 small triangles to *compose* a large triangle. Here are two ways to do so. If we place a large triangle on top of a composition of 4 small triangles and they match up exactly, we know that the area of the large triangle is equal to the combined area of 4 small triangles.



- We can *decompose* the large triangle into 4 small triangles. Again, we can reason that the area of one large triangle is equal to the combined area of 4 small triangles.



- Suppose we don't know the area of a small triangle, but we do know the area of a square that is composed of 2 small triangles. We can *decompose* the large triangle into 4 small triangles and then *rearrange* them into 2 squares. We can reason that the area of the large triangle is equal to the combined area of 2 squares. This is because when the 4 rearranged small triangles are placed on top of two squares, they match up exactly.



We will look more deeply into these strategies in the next lesson.

2.4 Tangram Rectangle

Cool Down: 5 minutes

Addressing

- 6.G.A.1

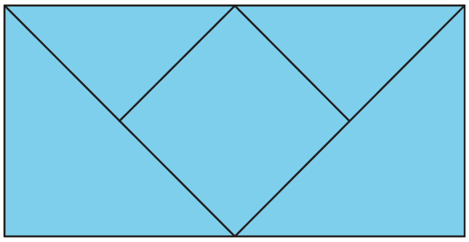
Launch

Give students access to the tangram shapes and geometry toolkits. Tell students that this figure is composed of two small right triangles, two medium right triangles, and a square, just like the ones they used earlier.

Note that students might not, at first, see the "square in the middle" as a square, or they might think of it a diamond (with unequal angles). Make sure that everyone understands that square-ness does not depend on how we turn the paper: A square is a rectangle (with all four angles being right angles) that has 4 equal sides.

Student Task Statement

The square in the middle has an area of 1 square unit. What is the area of the entire rectangle in square units? Explain your reasoning.



Student Response

The area is 4 square units. Possible strategies:

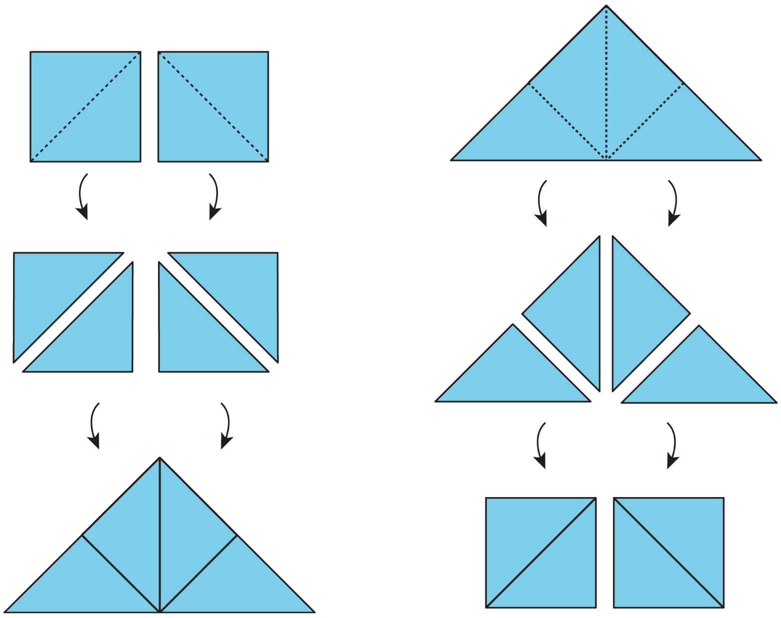
- Put together the two small triangles to make a square. Its area is 1 square unit. Decompose each medium triangle into two small triangles that can be arranged as a square. Each of these squares has area 1 square unit. Together with the square in the middle, the sum of the areas of these pieces is 4 square units.
- A small triangle has an area of $\frac{1}{2}$ square unit, and a medium triangle has an area of 1 square unit. $1 + 1 + 1 + \frac{1}{2} + \frac{1}{2} = 4$

Student Lesson Summary

Here are two important principles for finding **area**:

1. If two figures can be placed one on top of the other so that they match up exactly, then they have the *same area*.
2. We can **decompose** a figure (break a figure into pieces) and **rearrange** the pieces (move the pieces around) to find its area.

Here are illustrations of the two principles.



- Each square on the left can be decomposed into 2 triangles. These triangles can be rearranged into a large triangle. So the large triangle has the *same area* as the 2 squares.
- Similarly, the large triangle on the right can be decomposed into 4 equal triangles. The triangles can be rearranged to form 2 squares. If each square has an area of 1 square unit, then the area of the large triangle is 2 square units. We also can say that each small triangle has an area of $\frac{1}{2}$ square unit.

Glossary

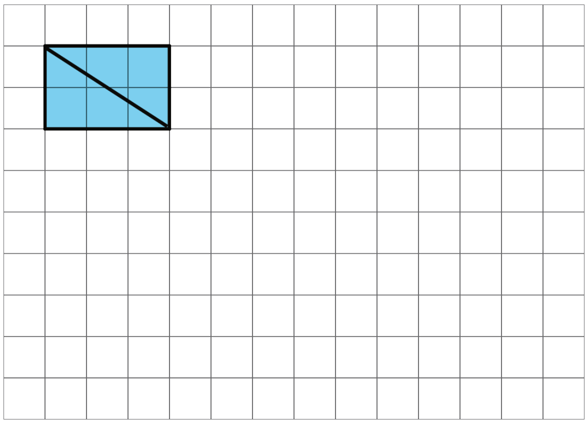
- compose
- decompose

Lesson 2 Practice Problems

Problem 1

Statement

The diagonal of a rectangle is shown.

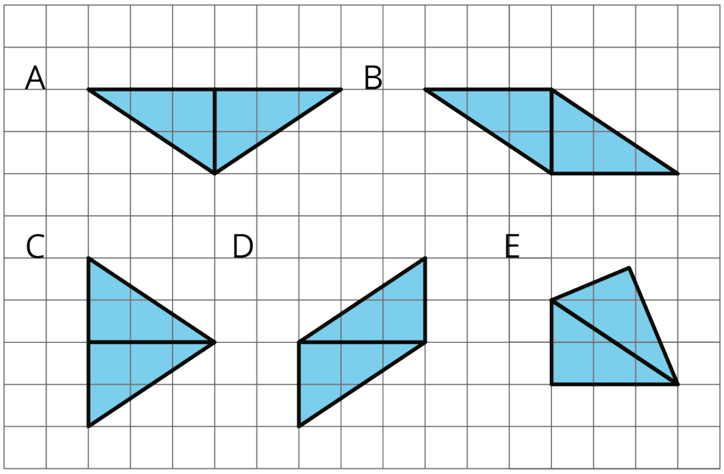


a. Decompose the rectangle along the diagonal, and recompose the two pieces to make a *different* shape.

b. How does the area of this new shape compare to the area of the original rectangle? Explain how you know.

Solution

a. Answers vary. Five different ways are shown.



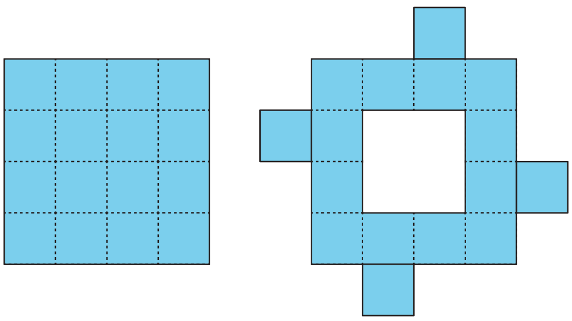
b. The areas are the same as all of the shapes are composed of two copies of the same triangle.

Problem 2

Statement

Priya decomposed a square into 16 smaller, equal-size squares and then cut out 4 of the small squares and attached them around the outside of original square to make a new figure.

How does the area of her new figure compare with that of the original square?



A. The area of the new figure is greater.

B. The two figures have the same area.

C. The area of the original square is greater.

D. We don’t know because neither the side length nor the area of the original square is known.

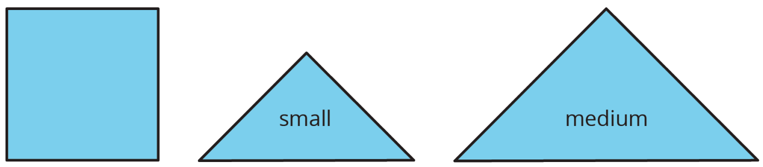
Solution

B

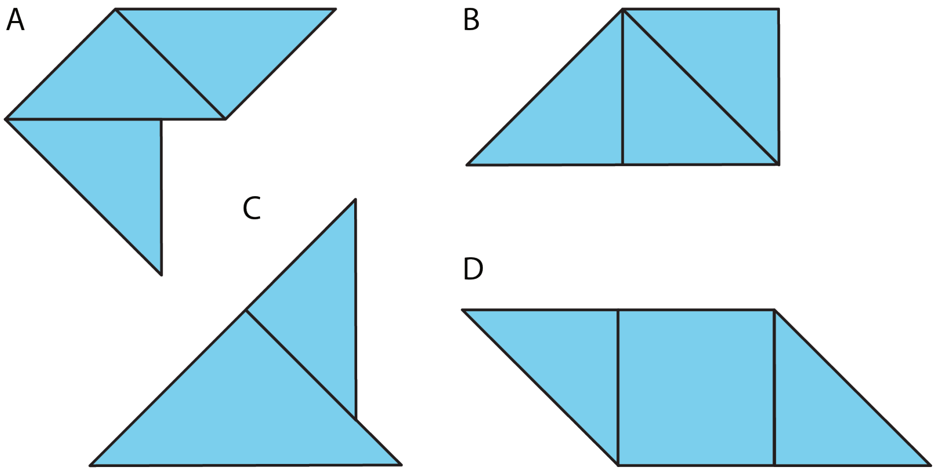
Problem 3

Statement

The area of the square is 1 square unit. Two small triangles can be put together to make a square or to make a medium triangle.



Which figure also has an area of $1\frac{1}{2}$ square units? Select **all** that apply.



A. Figure A

B. Figure B

C. Figure C

D. Figure D

Solution

["A", "B", "C"]

Problem 4

Statement

The area of a rectangular playground is 78 square meters. If the length of the playground is 13 meters, what is its width?

Solution

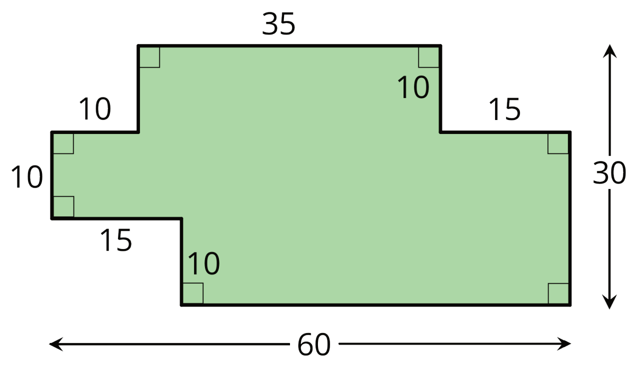
6 meters

(From Unit 1, Lesson 1.)

Problem 5

Statement

A student said, "We can't find the area of the shaded region because the shape has many different measurements, instead of just a length and a width that we could multiply."



Explain why the student's statement about area is incorrect.

Solution

Answers vary. Sample explanation: Area measures how many unit squares cover a region without gaps or overlaps. We multiply a length and a width when finding the area of a rectangle because that product tells us the number of unit squares in it. We can still find the area of a shape as shown, but first we will need to break it apart into rectangles whose areas we can find and then find the total area. We can also enclose the 30-by-60 region with a rectangle, find its area, and subtract the areas of the unshaded portions.

(From Unit 1, Lesson 1.)