Lesson 7: From Parallelograms to Triangles

Goals

- Describe (orally and in writing) ways in which two identical triangles can be composed, i.e., into a parallelogram or into a rectangle.
- Show how any parallelogram can be decomposed into two identical triangles by drawing a diagonal, and generalize (in writing) that this property applies to all parallelograms, but not all quadrilaterals.

Learning Targets

- I can explain the special relationship between a pair of identical triangles and a parallelogram.

Lesson Narrative

This lesson prepares students to apply what they know about the area of parallelograms to reason about the area of triangles.

Highlighting the relationship between triangles and parallelograms is a key goal of this lesson. The activities make use of both the idea of *decomposition* (of a quadrilateral into triangles) and *composition* (of two triangles into a quadrilateral). The two-way study is deliberate, designed to help students view and reason about the area of a triangle differently. Students see that a parallelogram can always be decomposed into two identical triangles, and that any two identical triangles can always be composed into a parallelogram (MP7).

Because a lot happens in this lesson and timing might be tight, it is important to both prepare all the materials and consider grouping arrangements in advance.

Alignments

Addressing

- 6.G.A.1: Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Building Towards

- 6.G.A.1: Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Instructional Routines

- MLR2: Collect and Display
- MLR8: Discussion Supports

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Pre-printed slips, cut from copies of the blackline master
Rulers

Required Preparation

Print pairs of triangles from the blackline master for A Tale of Two Triangles (Part 2). If students are cutting out the triangles, use the first page only. If the triangles are to be pre-cut by the teacher, print the second and third pages. Prepare enough sets so that each group of 3–4 students has a complete set (2 copies each of triangles P–U).

For classes using the digital version of the activity, an applet is provided that can be used in place of, or in addition to, the cut out triangles.

Student Learning Goals

Let’s compare parallelograms and triangles.

7.1 Same Parallelograms, Different Bases

Warm Up: 5 minutes
This warm-up reinforces students’ understanding of bases and heights in a parallelogram. In previous lessons, students calculated areas of parallelograms using bases and heights. They have also determined possible bases and heights of a parallelogram given a whole-number area. They saw, for instance, that finding possible bases and heights of a parallelogram with an area of 20 square units means finding two numbers with a product of 20. Students extend that work here by working with decimal side lengths and area.

As students work, notice students who understand that the two identical parallelograms have equal area and who use that understanding to find the unknown base. Ask them to share later.

Addressing

- 6.G.A.1

Launch

Give students 2 minutes of quiet work time and access to their geometry toolkits.

Students should be adequately familiar with bases and heights to begin the warm-up. If needed, however, briefly review the relationship between a pair of base and height in a parallelogram, using questions such as:

- "Can we use any side of a parallelogram as a base?" (Yes.)
- "Is the height always the length of one of the sides of the parallelogram?" (No.)
- "Once we have identified a base, how do we identify a height?" (It can be any segment that is perpendicular to the base and goes from the base to the opposite side.)
- "Can a height segment be drawn outside of a parallelogram?" (Yes.)

Anticipated Misconceptions

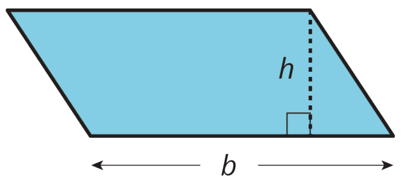
Some students may not know how to begin answering the questions because measurements are not shown on the diagrams. Ask students to label the parallelograms based on the information in the task statement.

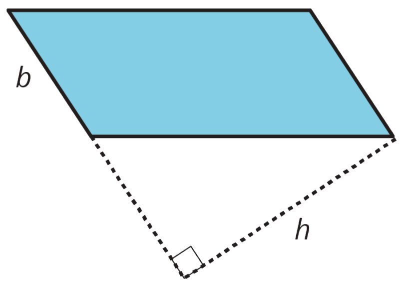
Students may say that there is not enough information to answer the second question because only one piece of information is known (the height). Ask them what additional information might be needed. Prompt them to revisit the task statement and see what it says about the two parallelograms. Ask what they know about the areas of two figures that are identical.

Students may struggle to find the unknown base in the second question because the area of the parallelogram is a decimal and they are unsure how to divide a decimal. Ask them to explain how they would reason about it if the area was a whole number. If they understand that they need to divide the area by 2 (since the height is 2 cm), see if they could reason in terms of multiplication (i.e., 2 times what number is 2.4?) or if they could reason about the division using fractions (i.e., 2.4 can be seen as $2\frac{4}{10}$ or $\frac{24}{10}$; what is 24 tenths divided by 2?).

Student Task Statement

Here are two copies of a parallelogram. Each copy has one side labeled as the base b and a segment drawn for its corresponding height and labeled h.





1. The base of the parallelogram on the left is 2.4 centimeters; its corresponding height is 1 centimeter. Find its area in square centimeters.

2. The height of the parallelogram on the right is 2 centimeters. How long is the base of that parallelogram? Explain your reasoning.

Student Response

1. The area of the first parallelogram is 2.4 square centimeters. $(2.4) \cdot 1 = 2.4$
2. The area of the second parallelogram is also 2.4 square centimeters. Since the base and height must multiply to the same area of 2.4, the base must be 1.2 centimeters because $(1.2) \cdot 2 = 2.4$.

Activity Synthesis

Select 1–2 previously identified students to share their responses. If not already explained by students, emphasize that we know the parallelograms have the same area because they are identical, which means that when one is placed on top of the other they would match up exactly.

Before moving on, ask students: “How can we verify that the height we found is correct, or that the two pairs of bases and heights produce the same area?" (We can multiply the values of each pair and see if they both produce 2.4.)

7.2 A Tale of Two Triangles (Part 1)

15 minutes (there is a digital version of this activity)
In earlier lessons, students saw that a square can be decomposed into two identical isosceles right triangles. They concluded that the area of each of those triangles is half of the area of the square. They used this observation to determine the area of composite regions.

This activity helps students see that parallelograms other than squares can also be decomposed into two identical triangles by drawing a diagonal. They check this by tracing a triangle on tracing paper and then rotating it to match the other copy. The process prepares students to see any triangle as occupying half of a parallelogram, and consequently, as having one half of its area. To generalize about quadrilaterals that can be decomposed into identical triangles, students need to analyze the features of the given shapes and look for structure (MP7).

There are a number of geometric observations in this unit that must be taken for granted at this point in students' mathematical study. This is one of those instances. Students have only seen examples of a parallelogram being decomposable into two copies of the same triangle, or have only verified this conjecture through physical experimentation, but for the time being it can be considered a fact. Starting in grade 8, they will begin to prove some of the observations they have previously taken to be true.

Building Towards

- 6.G.A.1

Instructional Routines

- MLR2: Collect and Display

Launch

Arrange students in groups of 3–4. Give students access to geometry toolkits and allow for 2 minutes of quiet think time for the first two questions. Then, ask them to share their drawings with their group and discuss how they drew their lines. If group members disagree on whether a quadrilateral can be decomposed into two identical triangles, they should note the disagreement, but it is not necessary to come to an agreement. They will soon have a chance to verify their responses.

Next, ask students to use tracing paper to check that the pairs of triangles that they believe to be identical are indeed so (i.e., they would match up exactly if placed on top of one another). Tell students to divide the checking work among the members of their group to optimize time.

Though students have worked with tracing paper earlier in the unit, some may not recall how to use it to check the congruence of two shapes; some explicit guidance might be needed. Encourage students to work carefully and precisely. A straightedge can be used in tracing but is not essential and may get in the way. Once students finish checking the triangles in their list and verify that they are identical (or correct their initial response), ask them to answer the last question.

Students using the digital activity can decompose the shapes using an applet. Encourage students to use the segment tool rather than free-drawing a segment to divide the shapes.

Access for Students with Disabilities

Representation: Internalize Comprehension. Represent the same information through different modalities by providing access to a hands-on alternative. To determine which quadrilaterals can be decomposed into two identical triangles, some students may benefit from enlarged cut-outs of the quadrilaterals that they can manipulate, fold, or cut.
Supports accessibility for: Conceptual processing; Visual-spatial processing

Access for English Language Learners

Conversing, Representing: MLR2 Collect and Display. To help students reason about and use the mathematical language of decompose, diagonal, and identical, listen to students talk about how they are making their drawings. Record and display common or important phrases you hear students say as well as examples of their drawings. Continue to update collected student language throughout the lesson, and remind students to borrow language from the display as needed.
Design Principle(s): Support sense-making; Maximize meta-awareness

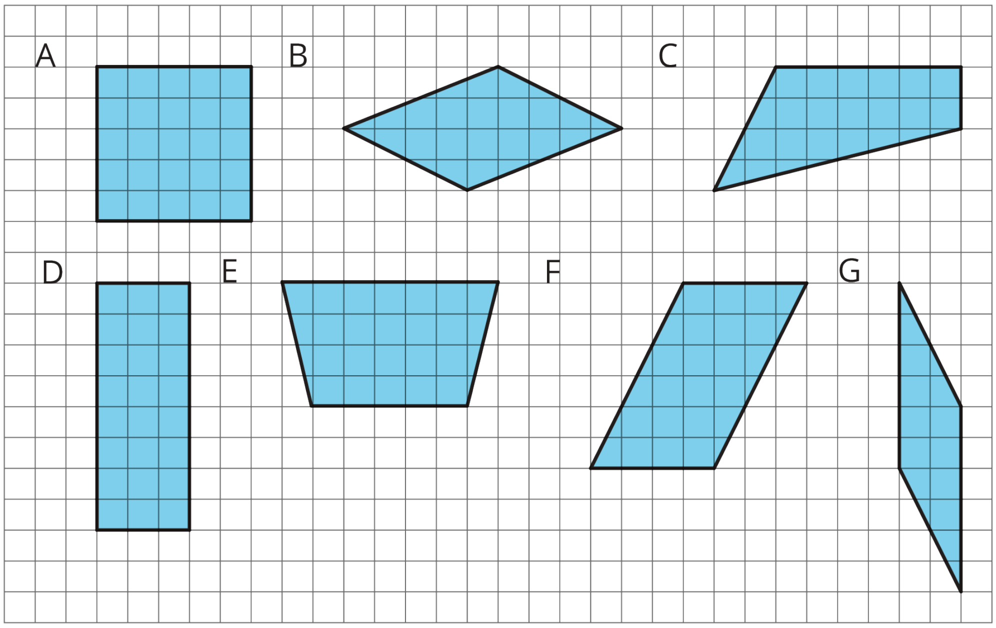
Anticipated Misconceptions

It may not occur to students to rotate triangles to check congruence. If so, tell students that we still consider two triangles identical even when one needs to be rotated to match the other.

Student Task Statement

Two polygons are identical if they match up exactly when placed one on top of the other.

1. Draw *one* line to decompose each polygon into two identical triangles, if possible. Use a straightedge to draw your line.



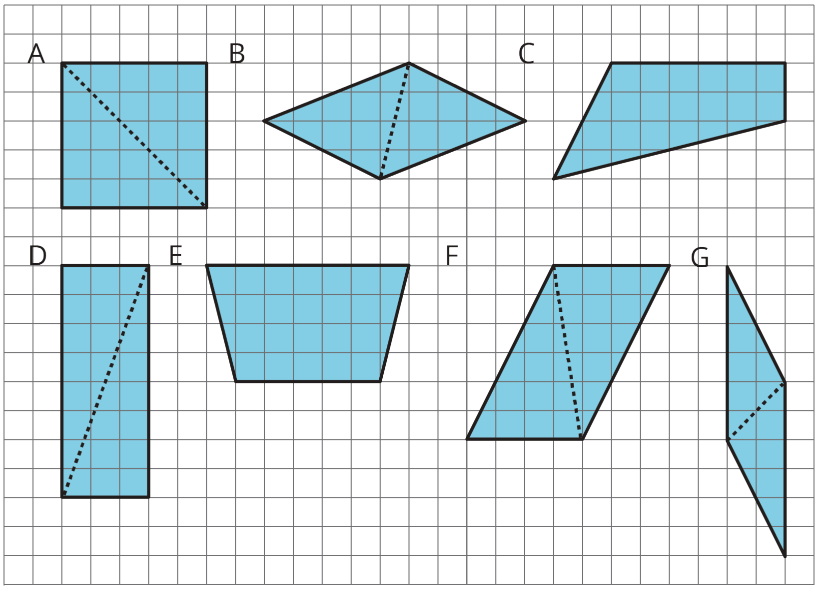
2. Which quadrilaterals can be decomposed into two identical triangles?

   Pause here for a small-group discussion.

3. Study the quadrilaterals that can, in fact, be decomposed into two identical triangles. What do you notice about them? Write a couple of observations about what these quadrilaterals have in common.

Student Response

1. Answers vary. Sample response:

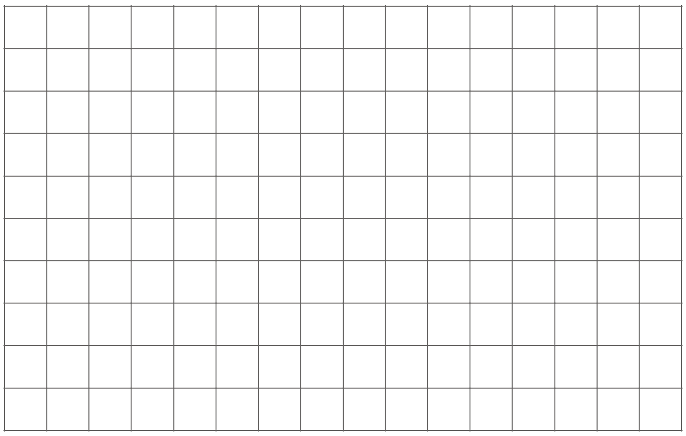


2. Answers vary. Quadrilaterals A, B, D, F, and G can be decomposed into two identical triangles.

3. Answers vary. Sample responses:
   - They have two pairs of parallel sides and each pair has equal length.
   - They are all parallelograms.
   - The triangles are formed by drawing a diagonal connecting opposite vertices.
   - Some triangles are right triangles, some are acute, and some are obtuse.
   - For some quadrilaterals, there is more than one way to decompose it into two identical triangles.

Are You Ready for More?

On the grid, draw some other types of quadrilaterals that are not already shown. Try to decompose them into two identical triangles. Can you do it?



> Come up with a rule about what must be true about a quadrilateral for it to be decomposed into two identical triangles.

Student Response

Answers vary.

Activity Synthesis

The discussion should serve two goals: to highlight how quadrilaterals can be decomposed into triangles and to help students make generalizations about the types of quadrilaterals that can be decomposed into two identical triangles. Consider these questions:

- How did you decompose the quadrilaterals into two triangles? (Connect opposite vertices, i.e. draw a diagonal.)
- Did the strategy of drawing a diagonal work for decomposing all quadrilaterals into two triangles? (Yes.) Are all of the resulting triangles identical? (No.)
- What is it about C and E that they cannot be decomposed into two identical triangles? (They don't have equal sides or equal angles. Their opposite sides are not parallel.)
- What do A, B, and D have that C and E do not? (A, B, and D have two pairs of parallel sides that are of equal lengths. They are parallelograms.)

Ask students to complete this sentence starter: For a quadrilateral to be decomposable into two identical triangles it must be (or have) . . .

If time permits, discuss how students verified the congruence of the two triangles.

- How did you check if the triangles are identical? Did you simply stack the traced triangle or did you do something more specific? (They may notice that it is necessary to rotate one triangle—or to reflect one triangle it twice—before the triangles could be matched up.)
- Did anyone use another way to check for congruence? (Students may also think in terms of the parts or composition of each triangle. E.g. "Both triangles have all the same side lengths; they both have a right angle").

7.3 A Tale of Two Triangles (Part 2)

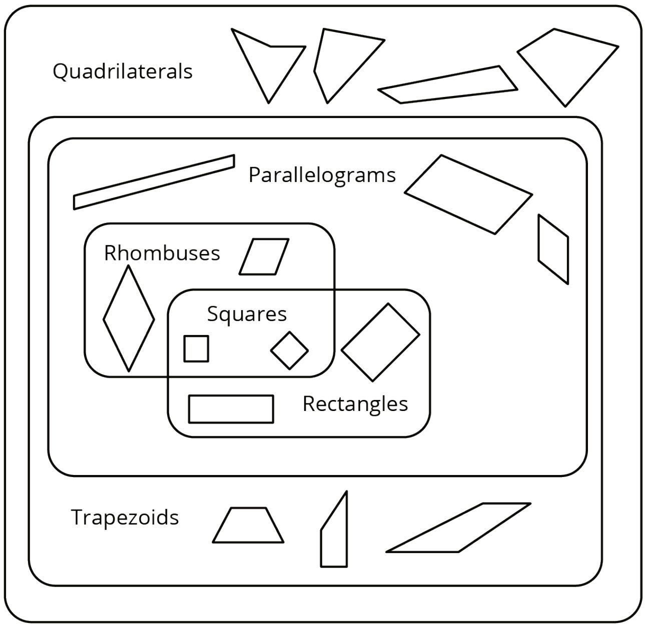
15 minutes (there is a digital version of this activity)

Previously, students decomposed quadrilaterals into two identical triangles. The work warmed them to the idea of a triangle as a *half* of a familiar quadrilateral. This activity prompts them to think the other way—to *compose* quadrilaterals using two identical triangles. It helps students see that two identical triangles of any kind can always be joined to produce a parallelogram. Both explorations prepare students to make connections between the area of a triangle and that of a parallelogram in the next lesson.

A key understanding to uncover here is that two identical copies of a triangle can be joined along any corresponding side to produce a parallelogram, and that more than one parallelogram can be formed.

As students work, look for different compositions of the same pair of triangles. Select students using different approaches to share later.

When manipulating the cutouts students are likely to notice that right triangles can be composed into rectangles (and sometimes squares) and that non-right triangles produce parallelograms that are not rectangles. Students may not immediately recall that squares and rectangles are also parallelograms. Consider preparing a reference for students to consult. Here is an example:



As before, students make generalizations here that they don't yet have the tools to justify them. This is appropriate at this stage. Later in their mathematical study they will learn to verify what they now take as facts.

For students using the digital activity, an applet can be used to compose triangles into other shapes.

Building Towards

- 6.G.A.1

Instructional Routines

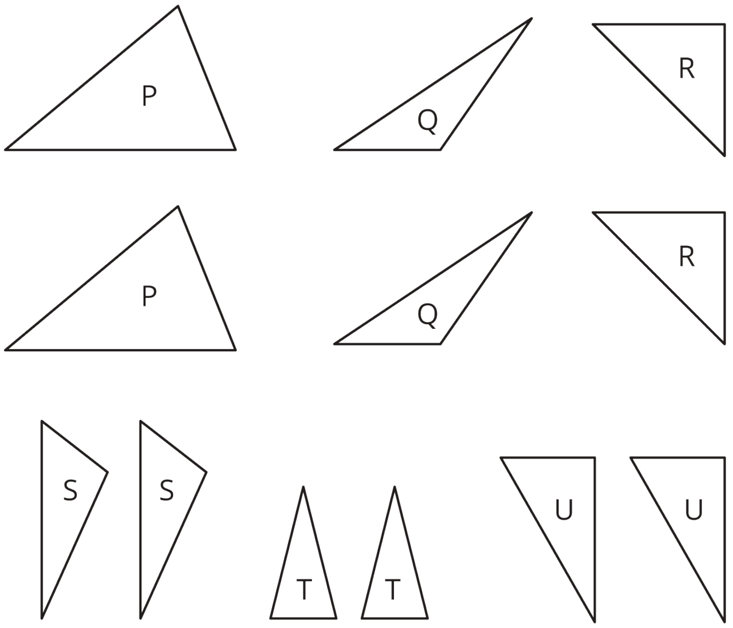
- MLR8: Discussion Supports

Launch

Keep students in the same groups. Give each group one set of triangles labeled P–U (two copies of each triangle) from the blackline master and access to scissors if the triangles are not pre-cut. The set includes different types of triangles (isosceles right, scalene right, obtuse, acute, and equilateral). Ask each group member to take 1–2 pairs of triangles.

Reiterate that students learned that certain types of quadrilaterals can be decomposed into two identical triangles. Explain that they will now see if it is possible to *compose* quadrilaterals out of two identical triangles, and, if so, to find out what types of quadrilaterals would result.

Give students 1–2 minutes of quiet work time, and then 5 minutes to discuss their responses and answer the second question with their group.



Access for English Language Learners

Conversing: MLR8 Discussion Supports. To reinforce use of the language that students have previously learned about quadrilaterals, create and display the reference chart, as described, for students to consult. Use this display to help students visualize the different types of quadrilaterals. Ask students to discuss with a partner, "What is the same and different about the different types of quadrilaterals?" Tell students to take turns sharing what they notice or remember from previous lessons, then call on different groups to share what they notice with the whole class. When recording and displaying students' observations, listen for opportunities to re-voice the mathematical terms that students used.
Design Principle(s): Support sense-making; Cultivate conversation

Anticipated Misconceptions

Students may draw incorrect conclusions if certain pieces of their triangles are turned over (to face down), or if it did not occur to them that the pieces could be moved. Ask them to try manipulating the pieces in different ways.

Seeing that two copies of a triangle can always be composed into a parallelogram, students might mistakenly conclude that any two copies of a triangle can *only* be composed into a parallelogram (i.e., no other quadrilaterals can be formed from joining two identical triangles). Showing a counterexample may be a simple way to help students see that this is not the case.

Student Task Statement

Your teacher will give your group several pairs of triangles. Each group member should take 1 or 2 pairs.

1. a. Which pair(s) of triangles do you have?

   b. Can each pair be composed into a rectangle? A parallelogram?

2. Discuss with your group your responses to the first question. Then, complete each statement with *All*, *Some*, or *None*. Sketch 1 or 2 examples to illustrate each completed statement.

   a. ______________ of these pairs of identical triangles can be composed into a *rectangle*.

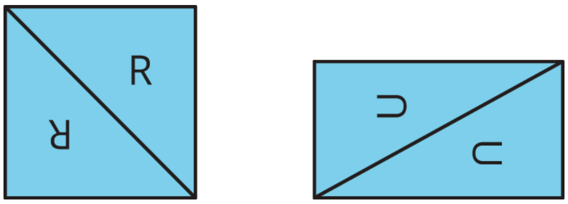
   b. ______________ of these pairs of identical triangles can be composed into a *parallelogram*.

Student Response

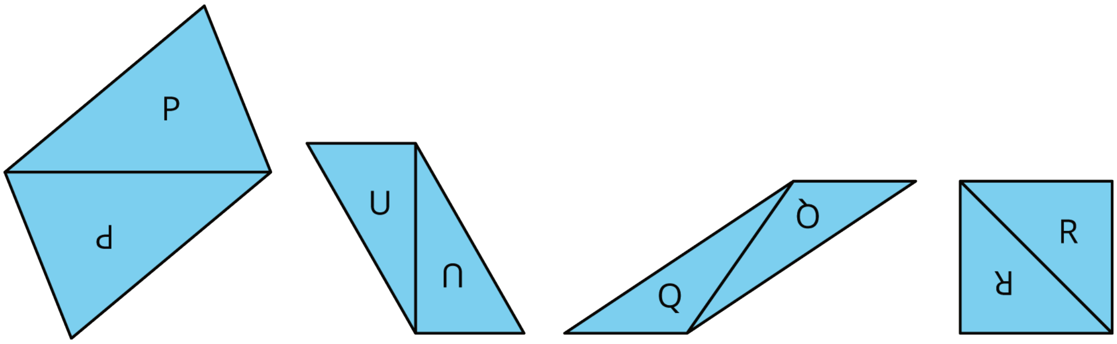
1. a. Answers vary. Yes for triangles R and U, no for the rest

   b. Yes for all triangles

2. a. *Some* of these pairs of triangles can be composed into a rectangle.



   b. *All* of these pairs of triangles can be composed into a *parallelogram*. Examples:



Activity Synthesis

The focus of this discussion would be to clarify whether or not two copies of each triangle can be composed into a rectangle or a parallelogram, and to highlight the different ways two triangles could be composed into a parallelogram.

Ask a few students who composed different parallelograms from the same pair of triangles to share. Invite the class to notice how these students ended up with different parallelograms. To help them see that a triangle can be joined along any side of its copy to produce a parallelogram, ask questions such as:

- Here is one way of composing triangles S into a parallelogram. Did anyone else do it this way? Did anyone obtain a parallelogram a different way?
- How many different parallelograms can be created with any two copies of a triangle? Why? (3 ways, because there are 3 sides along which the triangles could be joined.)
- What kinds of triangles can be used to compose a rectangle? How? (Right triangles, by joining two copies along the side opposite of the right angle.)
- What kinds of triangles can be used to compose a parallelogram? How? (Any triangle, by joining two copies along any side with the same length.)

Lesson Synthesis

Display and revisit representative works from the two main activities. Draw out key observations about the special connections between triangles and parallelograms.

First, we tried to decompose or break apart quadrilaterals into two identical triangles.

- “What strategy allowed us to do that?” (Drawing a segment connecting opposite vertices.)
- “Which types of quadrilaterals could always be decomposed into two identical triangles?” (Parallelograms.)
- “Can quadrilaterals that are not parallelograms be decomposed into triangles?” (Yes, but the resulting triangles may not be identical.)

Then, we explored the relationship between triangles and quadrilaterals the other way around. We tried to compose or create quadrilaterals from pairs of identical triangles.

- “What types of quadrilaterals were you able to compose with a pair of identical triangles?” (Parallelograms—some of them are rectangles.)
- “Does it matter what type of triangles was used?” (No. Any two copies of a triangle could be composed into a parallelogram.)
- “Was there a particular side along which the two triangles must be joined to form a parallelogram?” (No. Any of the three sides could be used.)

We saw how two identical copies of a triangle can be combined to make a parallelogram. This is true for any triangle. The reverse is also true: any parallelogram can be split into two identical triangles. In grade 8 we will acquire some tools to prove these observations. For now, we will take the special relationships between triangles and parallelograms as a fact. We will use them to find the area of any triangle in upcoming lessons.

7.4 A Tale of Two Triangles (Part 3)

Cool Down: 5 minutes

Building Towards

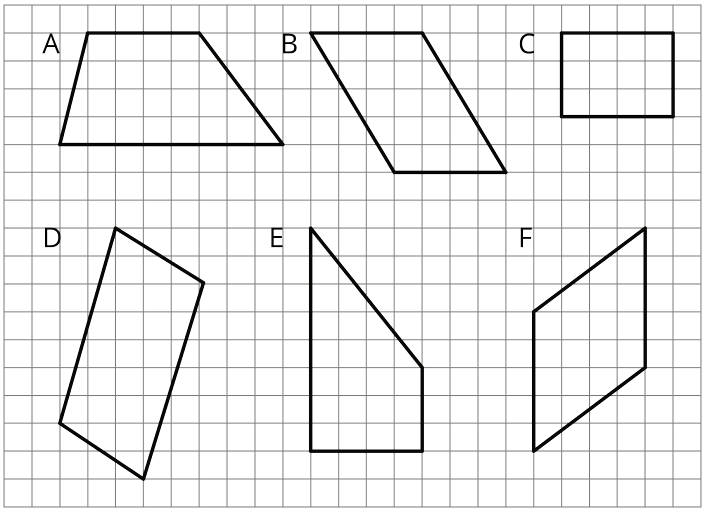
- 6.G.A.1

Launch

Give students access to their geometry toolkits if needed.

Student Task Statement

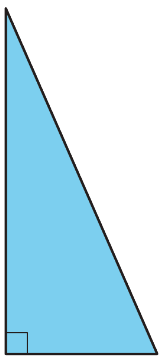
1. Here are some quadrilaterals.



   a. Circle all quadrilaterals that you think can be decomposed into two identical triangles using only one line.

   b. What characteristics do the quadrilaterals that you circled have in common?

2. Here is a right triangle. Show or briefly describe how two copies of it can be composed into a parallelogram.



Student Response

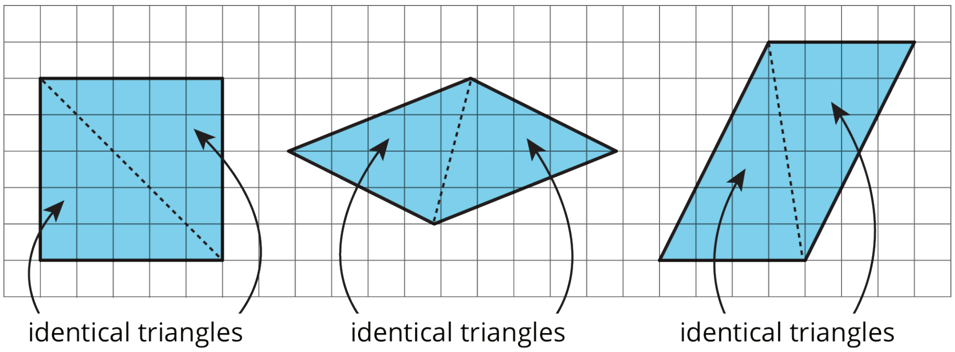
1. a. Quadrilaterals B, C, D, and F should be circled.

   b. They all have two pairs of parallel sides. They are all parallelograms.

2. Answers vary. Sample response: Joining two copies of the triangle along a side that is the same length (e.g., the shortest side of one and the shortest side of the other) would make a parallelogram. (Three parallelograms are possible, since there are three sides at which the triangles could be joined. One of the parallelograms is a rectangle.)

Student Lesson Summary

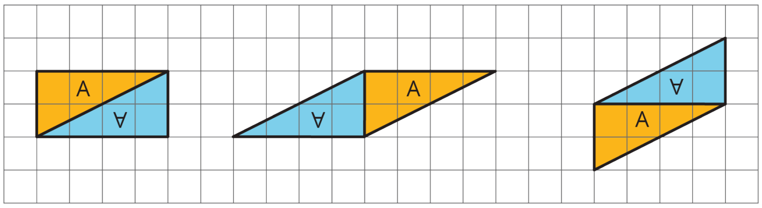
A parallelogram can always be decomposed into two identical triangles by a segment that connects opposite vertices.

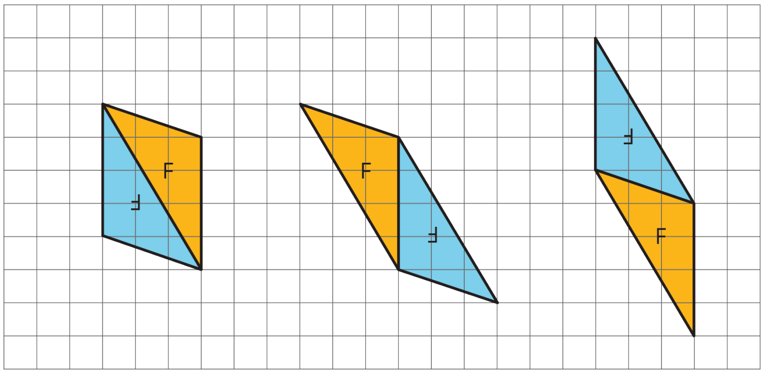


Going the other way around, two identical copies of a triangle can always be arranged to form a parallelogram, regardless of the type of triangle being used.

To produce a parallelogram, we can join a triangle and its copy along any of the three sides, so the same pair of triangles can make different parallelograms.

Here are examples of how two copies of both Triangle A and Triangle F can be composed into three different parallelograms.





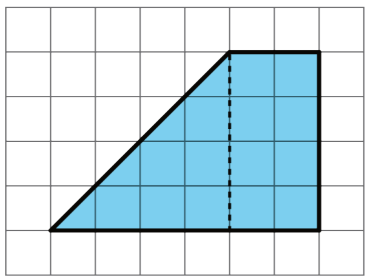
This special relationship between triangles and parallelograms can help us reason about the area of any triangle.

Lesson 7 Practice Problems

Problem 1

Statement

To decompose a quadrilateral into two identical shapes, Clare drew a dashed line as shown in the diagram.



a. She said the that two resulting shapes have the same area. Do you agree? Explain your reasoning.

b. Did Clare partition the figure into two identical shapes? Explain your reasoning.

Solution

a. Yes, the rectangle is 2 units by 4 units, so it has an area of 8 square units. The triangle is half of a 4-by-4 square, so its area is also 8 square units.

b. No, although the shapes have the same area, they are not identical shapes—one is a rectangle and the other a triangle.

Problem 2

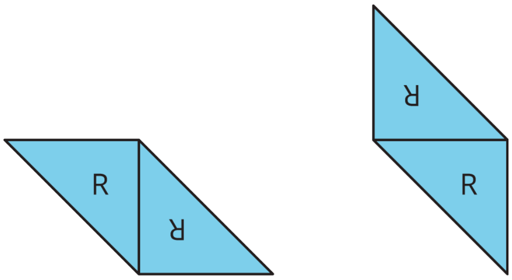
Statement

Triangle R is a right triangle. Can we use two copies of Triangle R to compose a parallelogram that is not a square?

If so, explain how or sketch a solution. If not, explain why not.

Solution

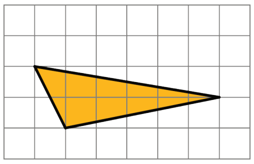
Yes, we can use two right triangles R to compose a parallelogram that is not a square by joining them along one of the shorter sides (the sides that make the right angle).

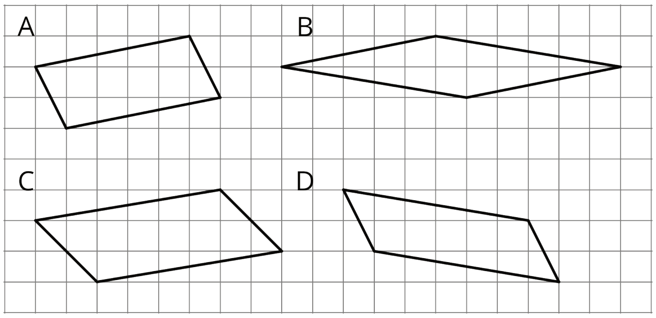


Problem 3

Statement

Two copies of this triangle are used to compose a parallelogram. Which parallelogram *cannot* be a result of the composition? If you get stuck, consider using tracing paper.





A. A

B. B

C. C

D. D

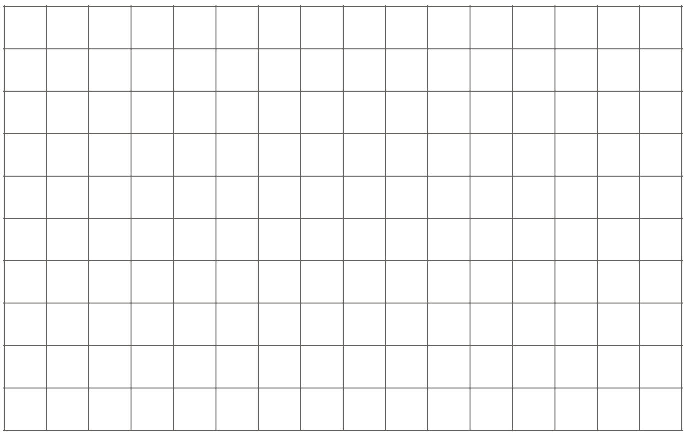
Solution

C

Problem 4

Statement

a. On the grid, draw at least three different quadrilaterals that can each be decomposed into two identical triangles with a single cut (show the cut line). One or more of the quadrilaterals should have non-right angles.

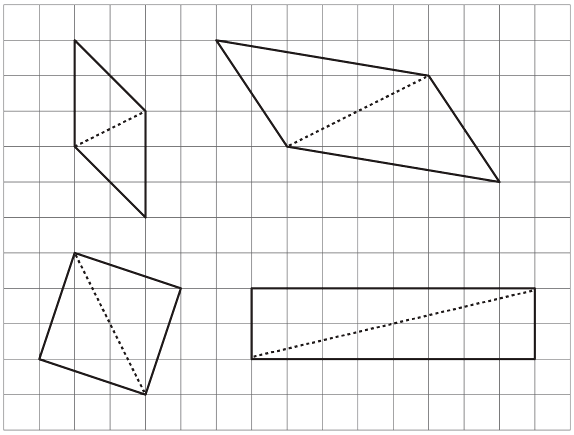


b. Identify the type of each quadrilateral.

Solution

Answers vary. Sample responses:

a.



b. The top two are parallelograms. The bottom left one is a square. The bottom right one is a rectangle. (All of them are parallelograms.)

Problem 5

Statement

a. A parallelogram has a base of 9 units and a corresponding height of $\frac{2}{3}$ units. What is its area?

b. A parallelogram has a base of 9 units and an area of 12 square units. What is the corresponding height for that base?

c. A parallelogram has an area of 7 square units. If the height that corresponds to a base is $\frac{1}{4}$ unit, what is the base?

Solution

a. $\frac{18}{3}$ square units (or equivalent)

b. $\frac{12}{9}$ units (or equivalent)

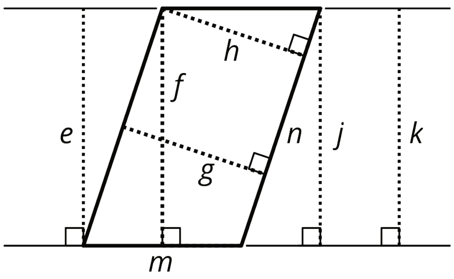
c. 28 units

(From Unit 1, Lesson 6.)

Problem 6

Statement

Select **all** the segments that could represent the height if side n is the base.



A. e

B. f

C. g

D. h

E. m

F. n

G. j

H. k

Solution

["C", "D"]

(From Unit 1, Lesson 5.)