## Lesson 21: Sums and Products of Rational and Irrational Numbers

* Let’s make convincing arguments about why the sums and products of rational and irrational numbers are always certain kinds of numbers.

### 21.1: Operations on Integers

Here are some examples of integers:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| -25 | -10 | -2 | -1 | 0 | 5 | 9 | 40 |

1. Experiment with adding any two numbers from the list (or other integers of your choice). Try to find one or more examples of two integers that:
	1. add up to another integer
	2. add up to a number that is *not* an integer
2. Experiment with multiplying any two numbers from the list (or other integers of your choice). Try to find one or more examples of two integers that:
	1. multiply to make another integer
	2. multiply to make a number that is *not* an integer

### 21.2: Sums and Products of Rational Numbers

1. Here are a few examples of adding two rational numbers. Is each sum a rational number? Be prepared to explain how you know.
	1. $4+0.175=4.175$
	2. $\frac{1}{2}+\frac{4}{5}=\frac{5}{10}+\frac{8}{10}=\frac{13}{10}$
	3. $-0.75+\frac{14}{8}=\frac{-6}{8}+\frac{14}{8}=\frac{8}{8}=1$
	4. $a$ is an integer: $\frac{2}{3}+\frac{a}{15}=\frac{10}{15}+\frac{a}{15}=\frac{10+a}{15}$
2. Here is a way to explain why the sum of two rational numbers is rational.
* Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are fractions. That means that $a,b,c,$ and $d$ are integers, and $b$ and $d$ are not 0.
	1. Find the sum of $\frac{a}{b}$ and $\frac{c}{d}$. Show your reasoning.
	2. In the sum, are the numerator and the denominator integers? How do you know?
	3. Use your responses to explain why the sum of $\frac{a}{b}+\frac{c}{d}$ is a rational number.
1. Use the same reasoning as in the previous question to explain why the product of two rational numbers, $\frac{a}{b}⋅\frac{c}{d}$, must be rational.

#### Are you ready for more?

Consider numbers that are of the form $a+b\sqrt{5}$, where $a$ and $b$ are integers. Let’s call such numbers *quintegers*.

Here are some examples of quintegers:

* $3+4\sqrt{5}$   ($a=3$, $b=4$)
* $7−2\sqrt{5}$   ($a=7$, $b=-2$)
* $-5+\sqrt{5}$   ($a=-5$, $b=1$)
* 3   ($a=3$, $b=0$).
1. When we add two quintegers, will we always get another quinteger? Either prove this, or find two quintegers whose sum is not a quinteger.
2. When we multiply two quintegers, will we always get another quinteger? Either prove this, or find two quintegers whose product is not a quinteger.

### 21.3: Sums and Products of Rational and Irrational Numbers

1. Here is a way to explain why $\sqrt{2}+\frac{1}{9}$ is irrational.
	* Let $s$ be the sum of $\sqrt{2}$ and $\frac{1}{9}$, or $s=\sqrt{2}+\frac{1}{9}$.
	* Suppose $s$ is rational.
	1. Would $s+-\frac{1}{9}$ be rational or irrational? Explain how you know.
	2. Evaluate $s+-\frac{1}{9}$. Is the sum rational or irrational?
	3. Use your responses so far to explain why $s$ cannot be a rational number, and therefore $\sqrt{2}+\frac{1}{9}$ cannot be rational.
2. Use the same reasoning as in the earlier question to explain why $\sqrt{2}⋅\frac{1}{9}$ is irrational.

### 21.4: Equations with Different Kinds of Solutions

1. Consider the equation $4x^{2}+bx+9=0$. Find a value of $b$ so that the equation has:
	1. 2 rational solutions
	2. 2 irrational solutions
	3. 1 solution
	4. no solutions
2. Describe all the values of $b$ that produce 2, 1, and no solutions.
3. Write a new quadratic equation with each type of solution. Be prepared to explain how you know that your equation has the specified type and number of solutions.
	1. no solutions
	2. 2 irrational solutions
	3. 2 rational solutions
	4. 1 solution

### Lesson 21 Summary

We know that quadratic equations can have rational solutions or irrational solutions. For example, the solutions to $\left(x+3\right)\left(x−1\right)=0$ are -3 and 1, which are rational. The solutions to $x^{2}−8=0$ are $\pm \sqrt{8}$, which are irrational.

Sometimes solutions to equations combine two numbers by addition or multiplication—for example, $\pm 4\sqrt{3}$ and $1+\sqrt{12}$. What kind of number are these expressions?

When we add or multiply two rational numbers, is the result rational or irrational?

* The sum of two rational numbers is rational. Here is one way to explain why it is true:
	+ Any two rational numbers can be written $\frac{a}{b}$ and $\frac{c}{d}$, where $a,b,c, and d$ are integers, and $b$ and $d$ are not zero.
	+ The sum of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ad+bc}{bd}$. The denominator is not zero because neither $b$ nor $d$ is zero.
	+ Multiplying or adding two integers always gives an integer, so we know that $ad,bc,bd$ and $ad+bc$ are all integers.
	+ If the numerator and denominator of $\frac{ad+bc}{bd}$ are integers, then the number is a fraction, which is rational.
* The product of two rational numbers is rational. We can show why in a similar way:
	+ For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, where $a,b,c, and d$ are integers, and $b$ and $d$ are not zero, the product is $\frac{ac}{bd}$.
	+ Multiplying two integers always results in an integer, so both $ac$ and $bd$ are integers, so $\frac{ac}{bd}$ is a rational number.

What about two irrational numbers?

* The sum of two irrational numbers could be either rational or irrational. We can show this through examples:
	+ $\sqrt{3}$ and $-\sqrt{3}$ are each irrational, but their sum is 0, which is rational.
	+ $\sqrt{3}$ and $\sqrt{5}$ are each irrational, and their sum is irrational.
* The product of two irrational numbers could be either rational or irrational. We can show this through examples:
	+ $\sqrt{2}$ and $\sqrt{8}$ are each irrational, but their product is $\sqrt{16}$ or 4, which is rational.
	+ $\sqrt{2}$ and $\sqrt{7}$ are each irrational, and their product is $\sqrt{14}$, which is not a perfect square and is therefore irrational.

What about a rational number and an irrational number?

* The sum of a rational number and an irrational number is irrational. To explain why requires a slightly different argument:
	+ Let $R$ be a rational number and $I$ an irrational number. We want to show that $R+I$ is irrational.
	+ Suppose $s$ represents the sum of $R$ and $I$ ($s=R+I$) and suppose $s$ is rational.
	+ If $s$ is rational, then $s+-R$ would also be rational, because the sum of two rational numbers is rational.
	+ $s+-R$ is not rational, however, because $\left(R+I\right)+-R=I$.
	+ $s+-R$ cannot be both rational and irrational, which means that our original assumption that $s$ was rational was incorrect. $s$, which is the sum of a rational number and an irrational number, must be irrational.
* The product of a non-zero rational number and an irrational number is irrational. We can show why this is true in a similar way:
	+ Let $R$ be rational and $I$ irrational. We want to show that $R⋅I$ is irrational.
	+ Suppose $p$ is the product of $R$ and $I$ ($p=R⋅I$) and suppose $p$ is rational.
	+ If $p$ is rational, then $p⋅\frac{1}{R}$ would also be rational because the product of two rational numbers is rational.
	+ $p⋅\frac{1}{R}$ is not rational, however, because $R⋅I⋅\frac{1}{R}=I$.
	+ $p⋅\frac{1}{R}$ cannot be both rational and irrational, which means our original assumption that $p$ was rational was false. $p$, which is the product of a rational number and an irrational number, must be irrational.



© CC BY 2019 by Illustrative Mathematics®