## Lesson 21: Sums and Products of Rational and Irrational Numbers

* Let’s make convincing arguments about why the sums and products of rational and irrational numbers are always certain kinds of numbers.

### 21.1: Operations on Integers

Here are some examples of integers:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| -25 | -10 | -2 | -1 | 0 | 5 | 9 | 40 |

1. Experiment with adding any two numbers from the list (or other integers of your choice). Try to find one or more examples of two integers that:
   1. add up to another integer
   2. add up to a number that is *not* an integer
2. Experiment with multiplying any two numbers from the list (or other integers of your choice). Try to find one or more examples of two integers that:
   1. multiply to make another integer
   2. multiply to make a number that is *not* an integer

### 21.2: Sums and Products of Rational Numbers

1. Here are a few examples of adding two rational numbers. Is each sum a rational number? Be prepared to explain how you know.
   1. is an integer:
2. Here is a way to explain why the sum of two rational numbers is rational.

* Suppose and are fractions. That means that and are integers, and and are not 0.
  1. Find the sum of and . Show your reasoning.
  2. In the sum, are the numerator and the denominator integers? How do you know?
  3. Use your responses to explain why the sum of is a rational number.

1. Use the same reasoning as in the previous question to explain why the product of two rational numbers, , must be rational.

#### Are you ready for more?

Consider numbers that are of the form , where and are integers. Let’s call such numbers *quintegers*.

Here are some examples of quintegers:

* (, )
* (, )
* (, )
* 3   (, ).

1. When we add two quintegers, will we always get another quinteger? Either prove this, or find two quintegers whose sum is not a quinteger.
2. When we multiply two quintegers, will we always get another quinteger? Either prove this, or find two quintegers whose product is not a quinteger.

### 21.3: Sums and Products of Rational and Irrational Numbers

1. Here is a way to explain why is irrational.
   * Let be the sum of and , or .
   * Suppose is rational.
   1. Would be rational or irrational? Explain how you know.
   2. Evaluate . Is the sum rational or irrational?
   3. Use your responses so far to explain why cannot be a rational number, and therefore cannot be rational.
2. Use the same reasoning as in the earlier question to explain why is irrational.

### 21.4: Equations with Different Kinds of Solutions

1. Consider the equation . Find a value of so that the equation has:
   1. 2 rational solutions
   2. 2 irrational solutions
   3. 1 solution
   4. no solutions
2. Describe all the values of that produce 2, 1, and no solutions.
3. Write a new quadratic equation with each type of solution. Be prepared to explain how you know that your equation has the specified type and number of solutions.
   1. no solutions
   2. 2 irrational solutions
   3. 2 rational solutions
   4. 1 solution

### Lesson 21 Summary

We know that quadratic equations can have rational solutions or irrational solutions. For example, the solutions to are -3 and 1, which are rational. The solutions to are , which are irrational.

Sometimes solutions to equations combine two numbers by addition or multiplication—for example, and . What kind of number are these expressions?

When we add or multiply two rational numbers, is the result rational or irrational?

* The sum of two rational numbers is rational. Here is one way to explain why it is true:
  + Any two rational numbers can be written and , where are integers, and and are not zero.
  + The sum of and is . The denominator is not zero because neither nor is zero.
  + Multiplying or adding two integers always gives an integer, so we know that and are all integers.
  + If the numerator and denominator of are integers, then the number is a fraction, which is rational.
* The product of two rational numbers is rational. We can show why in a similar way:
  + For any two rational numbers and , where are integers, and and are not zero, the product is .
  + Multiplying two integers always results in an integer, so both and are integers, so is a rational number.

What about two irrational numbers?

* The sum of two irrational numbers could be either rational or irrational. We can show this through examples:
  + and are each irrational, but their sum is 0, which is rational.
  + and are each irrational, and their sum is irrational.
* The product of two irrational numbers could be either rational or irrational. We can show this through examples:
  + and are each irrational, but their product is or 4, which is rational.
  + and are each irrational, and their product is , which is not a perfect square and is therefore irrational.

What about a rational number and an irrational number?

* The sum of a rational number and an irrational number is irrational. To explain why requires a slightly different argument:
  + Let be a rational number and an irrational number. We want to show that is irrational.
  + Suppose represents the sum of and () and suppose is rational.
  + If is rational, then would also be rational, because the sum of two rational numbers is rational.
  + is not rational, however, because .
  + cannot be both rational and irrational, which means that our original assumption that was rational was incorrect. , which is the sum of a rational number and an irrational number, must be irrational.
* The product of a non-zero rational number and an irrational number is irrational. We can show why this is true in a similar way:
  + Let be rational and irrational. We want to show that is irrational.
  + Suppose is the product of and () and suppose is rational.
  + If is rational, then would also be rational because the product of two rational numbers is rational.
  + is not rational, however, because .
  + cannot be both rational and irrational, which means our original assumption that was rational was false. , which is the product of a rational number and an irrational number, must be irrational.



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