Lesson 5: Circumference and Wheels

Goals

- Compare wheels of different sizes and explain (orally) why a larger wheel needs fewer rotations to travel the same distance.
- Generalize that the distance a wheel rolls in one rotation is equal to the circumference of the wheel.
- Write an equation to represent the proportional relationship between the number of rotations and the distance a wheel travels.

Learning Targets

- If I know the radius or diameter of a wheel, I can find the distance the wheel travels in some number of revolutions.

Lesson Narrative

This lesson is optional. The goal of this lesson is to apply students' understanding of circumference to calculate how far a wheel travels when it rolls a certain number of times. This relationship is vital for how odometers and speedometers work in vehicles.

In previous lessons, students saw that the relationships between radius, diameter, and circumference of different circles are proportional relationships. In this lesson, they notice that the circumference of a circle is the same as the distance a wheel rolls forward as it completes one rotation. Next, they see that there is also a proportional relationship between the number of times a wheel rotates and the distance the wheel travels. The last activity examines the relationship between the speed a vehicle is traveling and the number of rotations of the tires in a given amount of time.

Students make use of the structure of a proportional relationship as they work toward describing the relationship between the number of rotations of a wheel and the distance the wheel travels with an equation (MP7).

Alignments

Addressing

- 7.G.B.4: Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.
- 7.RP.A.2.a: Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

- 7.RP.A.2.c: Represent proportional relationships by equations. For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as $t = pn$.
- 7.RP.A.3: Use proportional relationships to solve multistep ratio and percent problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.

Instructional Routines

- MLR5: Co-Craft Questions
- MLR7: Compare and Connect
- MLR8: Discussion Supports
- Think Pair Share

Required Materials

Blank paper
Cylindrical household items
Receipt tape
Rulers

Required Preparation

You can reuse the same cylindrical household items from a previous lesson. Again, each group needs 3 items of relatively different sizes; however, it is not as important to include a wide variety of sizes. Because of the restrictions of paper size, you may want to forego using the larger objects (such as the paper plate) in this activity.

Prepare to distribute blank paper that is long enough for students to trace one complete rotation of their cylindrical object. For objects with a diameter greater than 4 inches, receipt tape may be better.

> ### Student Learning Goals
> Let’s explore how far different wheels roll.

5.1 A Rope and a Wheel

Warm Up: 5 minutes
This warm-up reminds students of the meaning and rough value of π. They apply this reasoning to a wheel and will continue to study wheels throughout this lesson. Students critique the reasoning of others (MP3).

Addressing

- 7.G.B.4

Launch

Give students 1 minute of quiet think time, 1 minute to discuss in small groups, then group discussion.

Student Task Statement

Han says that you can wrap a 5-foot rope around a wheel with a 2-foot diameter because $\frac{5}{2}$ is less than pi. Do you agree with Han? Explain your reasoning.

Student Response

Han is not correct. The circumference of the wheel is 2π feet. Since π is a little bit larger than 3, this is more than 6 feet, and the 5-foot rope will not fit all the way around.

Activity Synthesis

Ask students to share their reasoning emphasizing several important issues:

- The circumference of the wheel is 2π feet.
- π is larger than 3 (about 3.14) so the circumference of this wheel is more than 6 feet.
- Han is right that $\frac{5}{2} < \pi$, but this means that the rope will not make it all the way around.

Ask students if a 6-foot rope would be long enough to go around the wheel (no because $\frac{6}{2}$ is still less than π). What about a 7-foot rope? (Yes, because the circumference of the wheel is 2π feet and this is less than 7.)

Students may observe that it is possible to wrap the rope around the wheel going around a diameter twice as opposed to going around the circumference.

5.2 Rolling, Rolling, Rolling

Optional: 15 minutes
Students measure the circumference of circles by rolling them like wheels. They relate this to what they previously learned about the relationship between the diameter of a circle and its circumference. The circular objects that students measured earlier can be reused for this activity, the difference being that rather than wrapping something around each circle, they will roll the circle on a flat surface in order to measure its circumference. If reusing the same set of circular objects, make sure that the groups do not get the objects that they did in the previous activity.

Watch for students who realize that the relationship between the distance the circle rolls and the diameter is similar to the relationship between the circumference of a circle and the diameter. Encourage them to explain why this might be the case and ask them to share during the discussion.

Addressing

- 7.G.B.4

- 7.RP.A.2.a

Instructional Routines

- MLR7: Compare and Connect

Launch

Demonstrate rolling a circle along a straight line, marking where it starts and stops one complete rotation. Arrange students in groups of 3. Distribute 3 circular objects to each group. Provide access to blank paper (which should be long enough to complete one full rotation of each object) and rulers.

Student Task Statement

Your teacher will give you a circular object.

1. Follow these instructions to create the drawing:
    - On a separate piece of paper, use a ruler to draw a line all the way across the page.
    - Roll your object along the line and mark where it completes one rotation.
    - Use your object to draw tick marks along the line that are spaced as far apart as the diameter of your object.
2. What do you notice?
3. Use your ruler to measure each distance. Record these values in the first row of the table:
    a. the diameter of your object
    b. how far your object rolled in one complete rotation
    c. the quotient of how far your object rolled divided by the diameter of your object

object	diameter	distance traveled in one rotation	$\text{distance} \div \text{diameter}$

4. If you wanted to trace two complete rotations of your object, how long of a line would you need?
5. Share your results with your group and record their measurements in the table.

6. If each person in your group rolled their object along the entire length of the classroom, which object would complete the most rotations? Explain or show your reasoning.

Student Response

1. A line with 3 equally spaced tick marks and a little more length after the third tick mark.
2. The distance the circle rolled is a little more than three times the diameter.
3. Answers vary. Sample response: One row of the table filled in.

object	diameter	distance traveled in one rotation	$\text{distance} \div \text{diameter}$
soup can	6.8 cm	21.5 cm	3.2
tomato paste can	5.4 cm	17 cm	3.1
tuna can	8.5 cm	26.5 cm	3.1

4. Answers vary. Sample response: One rotation was about 21.5 cm so 2 rotations will require about 43 cm.
5. Answers vary. Sample response: Two more rows of the table filled in.
6. The smallest circular object, that is the one with the smallest diameter.

Activity Synthesis

The goal of this discussion is for students to understand that the distance a wheel travels in one complete rotation is equal to the circumference of the wheel.

Gather and display the quotients that students found in the table and emphasize that these values are close to π. Remind students of the activity from the other day when they measured the circumference of circular objects: "When we measure the circumference by wrapping a measuring tape around the circle, the circle stays in place while the measuring tape goes around it. When we roll the circle, we can imagine the measuring tape unwinding while the circle moves."

If desired, discuss which method of measuring the circumference was more precise (rolling the circle or wrapping a measuring tape around it)? Some reasons why measuring the circumference of the circle directly may be more precise include:

- When you roll the circular object, it is hard to keep it going in a straight line.
- It is difficult to mark one rotation precisely.

Access for English Language Learners

Representing, Conversing: MLR7 Compare and Connect. Once students are finished working, ask groups to choose a speaker who will remain at their seat and describe to visitors what was measured and discovered. Student groups will circulate the room, visiting each group. Provide students with the following format for speaking: show the three objects that were measured, describe the data found in your table, then explain what you discovered to the groups that visit. Listen for and amplify language that identifies measuring a rotation as being equal to measuring circumference. This routine will encourage conversation about explaining reasoning and methods for using rotations to find circumference.
Design Principle(s): Cultivate conversation; Maximize meta-awareness

5.3 Rotations and Distance

Optional: 15 minutes
In the previous activity, students saw that the distance a wheel travels in one rotation equals the circumference of the wheel. In this activity, students investigate proportional relationships between the number of rotations of a wheel and the distance that wheel travels. Students make repeated calculations with explicit numbers and then write an equation representing this proportional relationship (MP8).

Monitor for students who appropriately use their equation to answer the last part of each question, which also involves converting between inches and miles.

Addressing

- 7.G.B.4
- 7.RP.A.2.c

Instructional Routines

- MLR5: Co-Craft Questions
- Think Pair Share

Launch

Instruct students to use 3.14 as the approximation for π in these problems. Arrange students in groups of 3–4. Give students 6 minutes of quiet work time, 4 minutes of group discussion, followed by whole-class discussion.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Chunk this task into more manageable parts for students who benefit from support with organizational skills in problem solving. After the first 2–3 minutes of work time, invite 1–2 students to share their responses about the car wheel. Record their calculations on a display as they describe their reasoning.
Supports accessibility for: Organization; Attention

Access for English Language Learners

Writing, Conversing: MLR5 Co-craft Questions. Prior to revealing the task, show short video clips of a car wheel and a bicycle wheel rotating. Ask students to take a moment and think about what they have learned about "circumference" and "rotations" in previous lessons, and invite them to jot down at least two mathematical questions that could be asked about the video clips. Next, invite students to share their questions with a partner. Ask, "Do your questions share any language or ideas in common?", "Can you combine your ideas into one strong mathematical question the class could answer?" This helps students make comparisons between the sizes and types of wheel rotations they saw and how that applies to the task.
Design Principle(s): Cultivate conversation; Optimize output (for comparison)

Anticipated Misconceptions

Some students may struggle to convert between inches and miles for answering the last part of each question. Remind students that there are 5,280 feet in a mile. Ask students how many inches are in 1 foot. Make sure students arrive at a final answer of 63,360 inches in one mile before calculating the number of rotations made by each wheel.

Student Task Statement

1. A car wheel has a diameter of 20.8 inches.

    a. About how far does the car wheel travel in 1 rotation? 5 rotations? 30 rotations?

    b. Write an equation relating the distance the car travels in inches, c, to the number of wheel rotations, x.

    c. About how many rotations does the car wheel make when the car travels 1 mile? Explain or show your reasoning.

2. A bike wheel has a radius of 13 inches.

    a. About how far does the bike wheel travel in 1 rotation? 5 rotations? 30 rotations?

b. Write an equation relating the distance the bike travels in inches, b, to the number of wheel rotations, x.

c. About how many rotations does the bike wheel make when the bike travels 1 mile? Explain or show your reasoning.

Student Response

1. a. In one revolution, the car travels 20.8π inches or about 65.3 inches. In 5 revolutions, the car will travel 5 times as far or about 327 inches. In 30 revolutions of the wheel, the car will travel $30 \cdot (20.8\pi)$ feet or about or about 1,960 inches.

   b. In x revolutions of the wheel, the car travels $20.8\pi x$ inches, $c = 20.8\pi x$.

   c. One mile is 5,280 feet or 63,360 inches. Dividing this by 20.8π will give the number of wheel revolutions to go one mile, about 970.

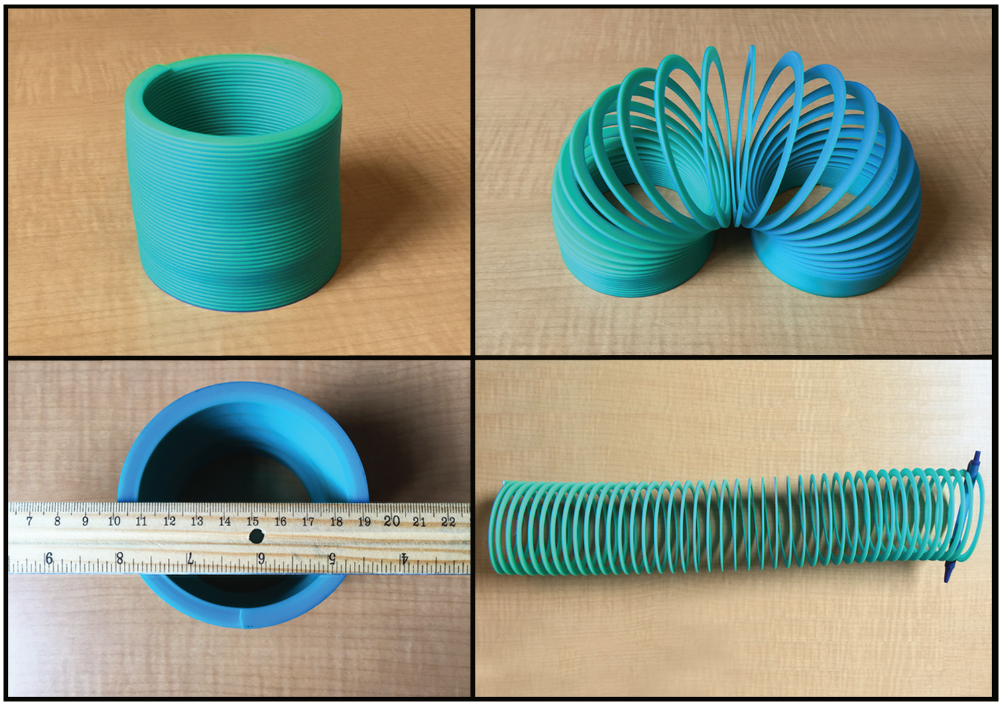
2. a. In one revolution, the bike travels 26π inches or about 81.7 inches. In 5 revolutions, the bike will travel 5 times as far or about 408 inches. In 30 revolutions of the wheel, the bike will travel $30 \cdot (26\pi)$ feet or about or about 2,450 inches.

   b. In x revolutions of the wheel, the bike travels $26\pi x$ inches, $b = 26\pi x$.

   c. One mile is 5,280 feet or 63,360. Dividing this by 26π will give the number of wheel revolutions to go one mile, about 776.

Are You Ready for More?

Here are some photos of a spring toy.



> If you could stretch out the spring completely straight, how long would it be? Explain or show your reasoning.

Student Response

We can compute the approximate length if we know the diameter of the circle and the total number of loops. Diameter is ≈ 9.5 cm. There are 42 loops. Therefore, $(9.5)(\pi)(42) \approx 1252.86$. The length is about 1,253 cm, or about 12.5 meters.

Activity Synthesis

The majority of the discussion will occur in small groups, but here are some things to debrief with the whole class:

- "What is the constant of proportionality for each relationship? What does that tell you about the situation?" (65.3 and 81.7, the number of inches each wheel travels per rotation, which is also the circumference of each wheel.)
- "How do the two wheels compare? How can you see this in the equations?" (The bike wheel is larger. The constant of proportionality is larger in the equation representing the bike.)

Poll the class on which wheel makes fewer rotations to travel one mile (the bike). Invite students to explain why. (Its wheels are larger, so it moves farther in one rotation.)

5.4 Rotations and Speed

Optional: 15 minutes
In the previous activity, students relate the distance a wheel travels to the number of rotations the wheel has made. This activity introduces a new quantity, the *speed* the wheel travels. As long as the rate the wheel rotates does not change, there is a nice relationship between the distance the wheel travels, d, and the amount of time, t. With appropriate units, $d = rt$: here r is the speed the wheel travels, which can be calculated in terms of the rate at which the wheel spins. The goal of this activity is to develop and explore this relationship.

As they work on this activity, students will observe repeatedly that the distance the wheel travels is determined by the amount of time elapsed and the rate at which the wheel spins.

Watch for students who use the calculations that they have made in earlier problems as a scaffold for answering later questions. For example, when the car wheels rotate once per second, the car travels about 3.7 mph. When the wheels rotate 5 times per second, that means that the car will travel 5 times as fast or about 18.5 miles per hour. Ask these students to share their work during the discussion.

Addressing

- 7.G.B.4
- 7.RP.A.3

Instructional Routines

- MLR8: Discussion Supports

Launch

Remind students to be careful with units as they work through the problems. Also remind them that there are 5,280 feet in a mile and 12 inches in a foot.

Anticipated Misconceptions

Some students may do the calculations in feet but not know how to convert their answers to miles. Remind them that there are 5,280 feet in 1 mile.

Student Task Statement

The circumference of a car wheel is about 65 inches.

1. If the car wheel rotates once per second, how far does the car travel in one minute?
2. If the car wheel rotates once per second, about how many miles does the car travel in one hour?
3. If the car wheel rotates 5 times per second, about how many miles does the car travel in one hour?
4. If the car is traveling 65 miles per hour, about how many times per second does the wheel rotate?

Student Response

1. There are 60 seconds in a minute so if the car wheel rotates once per second, that's 60 rotations in a minute. At 65 inches per rotation that is 3,900 inches ($60 \boldsymbol{\cdot} 65$). That's the same as 325 feet.
2. There are 60 minutes in an hour, and the car travels 325 feet in a minute so in one hour the car would travel 19,500 feet ($60 \boldsymbol{\cdot} 325$) if the wheels rotate once per second. This is about 3.7 miles, because $19{,}500 \div 5{,}280 \approx 3.7$.
3. If the car wheel rotates 5 times per second, then the car will travel 5 times farther than it did when they rotated once per second. So that's about 97,500 feet ($5 \boldsymbol{\cdot} 19500$). This is about 18.5 miles, because $97{,}500 \div 5{,}280 \approx 18.5$.
4. If the car travels 65 miles per hour that is 343,200 feet in an hour. Each rotation of the wheel per second amounts to 19,500 feet traveled in one hour. So to travel 343,200 feet in an hour the wheels will rotate $343200 \div 19500$ times per second. That is about 17.6 times.

Activity Synthesis

Have students share their solutions to the first question and emphasize the different ratios and rates that come up when solving this problem. There are

- 65 inches of distance traveled *per* rotation of the wheels

- 1 rotation of the wheel *per* second
- 60 seconds *per* minute

Finally, there are 12 inches *per* foot if students convert the answer to feet. While this is not essential, it is useful in the next problem as feet are a good unit for expressing both inches and miles.

For the last problem, different solutions are possible including:

- In the second question, students have found that the car travels 19,500 feet in an hour if the wheels rotate once per second. This is about 3.7 miles. The number of rotations per second is proportional to the speed. So one way to find how many times the wheels rotate at 65 miles per hour would be to calculate $65 \div 3.7$.
- Students could similarly use their answer to the third question, checking what multiple of 18.5 miles (the distance traveled in an hour at 5 rotations per second) gives 65.
- A calculation can be done from scratch, checking how many inches are in 65 miles, how many seconds are in an hour, and then finding how many rotations of the wheel per second result in traveling 65 miles.

The first of these approaches should be stressed in the discussion as it uses the previous calculations students have made in an efficient way.

Access for English Language Learners

Conversing: MLR8 Discussion Supports. To help students to compare and justify their methods for answering the problem about the car traveling 65 mph, ask them to compare their responses with a partner. Listen for and amplify any language that describes different methods for calculations. Invite students to zero in on "why" there are more differences here. Provide students with question stems to help them compare and contrast, such as: "Do you have the same answer? If not, then why?" (Students may have estimated or converted differently or incorrectly.), "Did you use the same conversions for units?", and "Why do your answers differ?"
Design Principle(s): Maximize meta-awareness; Cultivate conversation

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Lesson Synthesis

Key points in this lesson include:

- The circumference of a circle is how far the circle rolls in one complete revolution.
- If d is the distance a circular wheel rolls in x rotations, then $d = Cx$ where C is the circumference of the wheel.

5.5 Biking Distance

Cool Down: 5 minutes

Addressing

- 7.G.B.4
- 7.RP.A.3

Student Task Statement

The wheels on Noah's bike have a circumference of about 5 feet.

1. How far does the bike travel as the wheel makes 15 complete rotations?
2. How many times do the wheels rotate if Noah rides 40 feet?

Student Response

1. 75 feet, because $5 \cdot 15 = 75$
2. 8 rotations, because $40 \div 5 = 8$

Student Lesson Summary

The circumference of a circle is the distance around the circle. This is also how far the circle rolls on flat ground in one rotation. For example, a bicycle wheel with a diameter of 24 inches has a circumference of 24π inches and will roll 24π inches (or 2π feet) in one complete rotation. There is an equation relating the number of rotations of the wheel to the distance it has traveled. To see why, let's look at a table showing how far the bike travels when the wheel makes different numbers of rotations.

number of rotations	distance traveled (feet)
1	2π
2	4π
3	6π
10	20π
50	100π
x	?

In the table, we see that the relationship between the distance traveled and the number of wheel rotations is a proportional relationship. The constant of proportionality is 2π.

To find the missing value in the last row of the table, note that each rotation of the wheel contributes 2π feet of distance traveled. So after x rotations the bike will travel $2\pi x$ feet. If d is the distance, in feet, traveled when this wheel makes x rotations, we have the relationship:

$$d = 2\pi x$$

Lesson 5 Practice Problems

Problem 1

Statement

The diameter of a bike wheel is 27 inches. If the wheel makes 15 complete rotations, how far does the bike travel?

Solution

$405 \boldsymbol{\cdot} \pi$ or about 1,272 inches (106 feet)

Problem 2

Statement

The wheels on Kiran's bike are 64 inches in circumference. How many times do the wheels rotate if Kiran rides 300 yards?

Solution

About 169 times. There are 36 inches in a yard so 10,800 inches in 300 yards and $10{,}800 \div 64 \approx 169$.

Problem 3

Statement

The numbers are measurements of radius, diameter, and circumference of circles A and B. Circle A is smaller than circle B. Which number belongs to which quantity?

2.5, 5, 7.6, 15.2, 15.7, 47.7

Solution

Circle A: radius 2.5, diameter 5, circumference 15.7 Circle B: radius 7.6, diameter 15.2, circumference 47.7

(From Unit 3, Lesson 4.)

Problem 4

Statement

Circle A has circumference $2\frac{2}{3}$ m. Circle B has a diameter that is $1\frac{1}{2}$ times as long as Circle A’s diameter. What is the circumference of Circle B?

Solution

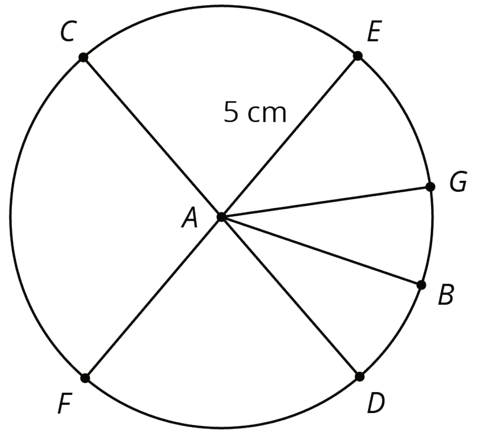
4 m. If the diameter of Circle B is $1\frac{1}{2}$ times larger than Circle A, its circumference must be as well. We can rewrite to calculate: $\left(\frac{8}{3}\right)\left(\frac{3}{2}\right) = 4$.

(From Unit 3, Lesson 3.)

Problem 5

Statement

The length of segment AE is 5 centimeters.



a. What is the length of segment CD?

b. What is the length of segment AB?

c. Name a segment that has the same length as segment AB.

Solution

a. 10 cm

b. 5 cm

c. Answers vary. Sample responses: CA, AF, AD, AG, AE

(From Unit 3, Lesson 2.)