Lesson 15: The Remainder Theorem

- Let's learn about the Remainder Theorem.

15.1: Notice and Wonder: Division Leftovers

What do you notice? What do you wonder?

$$\begin{array}{r} 33 \\ 10\overline{)330} \\ \underline{300} \\ 30 \\ \underline{30} \\ 0 \end{array} \qquad \begin{array}{r} 82 \\ 4\overline{)330} \\ \underline{320} \\ 10 \\ \underline{8} \\ 2 \end{array} \qquad \begin{array}{r} 66 \\ 5\overline{)330} \\ \underline{300} \\ 30 \\ \underline{30} \\ 0 \end{array}$$

A. $330 = 33(10) + 0$

B. $330 = 4(82) + 2$

C. $330 = 5(66) + 0$

15.2: The Unknown Coefficient

Consider the polynomial function $f(x) = x^4 - ux^3 + 24x^2 - 32x + 16$ where u is an unknown real number. If $x - 2$ is a factor, what is the value of u? Explain how you know.

Are you ready for more?

Here are some diagrams that show the same third-degree polynomial, $P(x) = 2x^3 + 5x^2 + x + 10$, divided by a linear factor and by a quadratic factor.

$$\frac{P(x)}{x+3}$$

	$2x^2$	$-x$	4
x	$2x^3$	$-x^2$	$4x$
3	$6x^2$	$-3x$	12

$$\frac{P(x)}{x^2 - x}$$

	$2x$	7
x^2	$2x^3$	$7x^2$
$-x$	$-2x^2$	$-7x$

1. What is the remainder of each of these divisions?

2. For each division, how does the degree of the remainder compare to the degree of the divisor?

3. Could the remainder ever have the same degree as the divisor, or a higher degree? Give an example to show that this is possible, or explain why it is not possible.

15.3: A Study of Remainders

1. Which of these polynomials could have $(x-2)$ as a factor?
    a. $A(x)=6x^2-7x-5$

    b. $B(x)=3x^2+15x-42$

    c. $C(x)=2x^3+13x^2+16x+5$

    d. $D(x)=3x^3-2x^2-15x+14$

    e. $E(x)=8x^4-41x^3-18x^2+101x+70$

    f. $F(x)=x^4+5x^3-27x^2-101x-70$

2. Select one of the polynomials that you said doesn't have $(x-2)$ as a factor.
    a. Explain how you know $(x-2)$ is not a factor.

    b. If you have not already done so, divide the polynomial by $(x-2)$. What is the remainder?

3. List the remainders for each of the polynomials when divided by $(x-2)$. How do these values compare to the value of the functions at $x=2$?

Lesson 15 Summary

When we use long division to divide 1573 by 12, we get a remainder of 1, so $1573 = 12(131) + 1$. When we divide by 11 instead, we get a remainder of 0, so $1573 = 11(143)$. A remainder of 0 means that 11 is a factor of 1573. The same thing happens with polynomials. While $(x^3 + 5x^2 + 7x + 3) \div (x + 2)$ results in a remainder that is not 0, if we divide $(x + 1)$ into $x^3 + 5x^2 + 7x + 3$, we do get a remainder of 0:

$$
\begin{array}{r@{}l}
 & x^2 + 4x + 3 \\
x + 1 \big) & \overline{x^3 + 5x^2 + 7x + 3} \\
 & \underline{-x^3 - x^2} \\
 & \quad\;\; 4x^2 + 7x \\
 & \quad\;\; \underline{-4x^2 - 4x} \\
 & \qquad\qquad\;\; 3x + 3 \\
 & \qquad\qquad\;\; \underline{-3x - 3} \\
 & \qquad\qquad\qquad\;\; 0
\end{array}
$$

So $(x + 1)$ is a factor of $x^3 + 5x^2 + 7x + 3$.

Earlier we learned that if $(x - a)$ is a factor of a polynomial $p(x)$, then $p(a) = 0$, meaning a is a zero of the function. It turns out that the converse is also true: if a is a zero, then $(x - a)$ is a factor.

To see that this is true, let's think about what we know if we have a polynomial $p(x)$ with a known zero at $x = a$. If we divide $p(x)$ by the linear factor $(x - a)$, then $p(x) = (x - a)q(x) + r$, where r is the remainder and $q(x)$ is a polynomial. Because a is a zero of the function, we know that $p(a) = 0$. This means we also know that the remainder is zero:

$$
\begin{aligned}
p(a) &= (a - a)q(x) + r \\
p(a) &= r \\
0 &= r
\end{aligned}
$$

Which means that $p(x) = (x - a)q(x)$. So, if a is a zero of a polynomial, then $(x - a)$ must be a factor of $p(x)$. Now we know that if we start with a linear factor of a polynomial, then we know one of the zeros of the polynomials, and if we start with a zero of a polynomial, then we know one of the linear factors.

Lastly, even if a is not a zero of p, we can figure out what the remainder will be if we divide $p(x)$ by $(x - a)$, without having to do any division. If $p(x) = (x - a)q(x) + r$, then $p(a) = (a - a)q(x) + r$, so $p(a) = r$. So the remainder after division by $(x - a)$ is $p(a)$. This is the Remainder Theorem.