Lesson 9: Moves in Parallel

Goals

- Comprehend that a rotation by 180 degrees about a point of two intersecting lines moves each angle to the angle that is vertical to it.
- Describe (orally and in writing) observations of lines and parallel lines under rigid transformations, including lines that are taken to lines and parallel lines that are taken to parallel lines.
- Draw and label rigid transformations of a line and explain the relationship between a line and its image under the transformation.
- Generalize (orally) that "vertical angles" are congruent using informal arguments about 180 degree rotations of lines.

Learning Targets

- I can describe the effects of a rigid transformation on a pair of parallel lines.
- If I have a pair of vertical angles and know the angle measure of one of them, I can find the angle measure of the other.

Lesson Narrative

The previous lesson examines the impact of rotations on line segments and polygons. This lesson focuses on the effects of rigid transformations on lines. In particular, students see that parallel lines are taken to parallel lines and that a 180° rotation about a point on the line takes the line to itself. In grade 7, students found that **vertical angles** have the same measure, and they justify that here using a 180° rotation.

As they investigate how 180° rotations influence parallel lines and intersecting lines, students are looking at specific examples but their conclusions hold for all pairs of parallel or intersecting lines. No special properties of the two intersecting lines are used so the 180° rotation will show that vertical angles have the same measure for *any* pair of vertical angles.

Alignments

Building On

- 7.G.B.5: Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.

Addressing

- 8.G.A.1.a: Lines are taken to lines, and line segments to line segments of the same length.
- 8.G.A.1.b: Angles are taken to angles of the same measure.

- 8.G.A.1.c: Parallel lines are taken to parallel lines.

Instructional Routines

- MLR7: Compare and Connect

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Tracing paper
Bundles of "patty paper" are available commercially for a very low cost. These are small sheets (about 5" by 5") of transparent paper.

Student Learning Goals

Let's transform some lines.

9.1 Line Moves

Warm Up: 10 minutes
In this warm-up, students continue their work with transformations by shifting from applying rigid transformations to shapes to applying them specifically to lines. Each image in this activity has the same starting line and students are asked to name the translation, rotation, or reflection that takes this line to the second marked line. Because of their infinite and symmetric nature, different transformations of lines look the same unless specific points are marked, so 1–2 points on each line are marked.

While students have experience transforming a variety of figures, this activity provides the opportunity to use precise language when describing transformations of lines while exploring how sometimes different transformations can result in the same final figures. During the activity, encourage students to look for more than one way to transform the original line.

Addressing

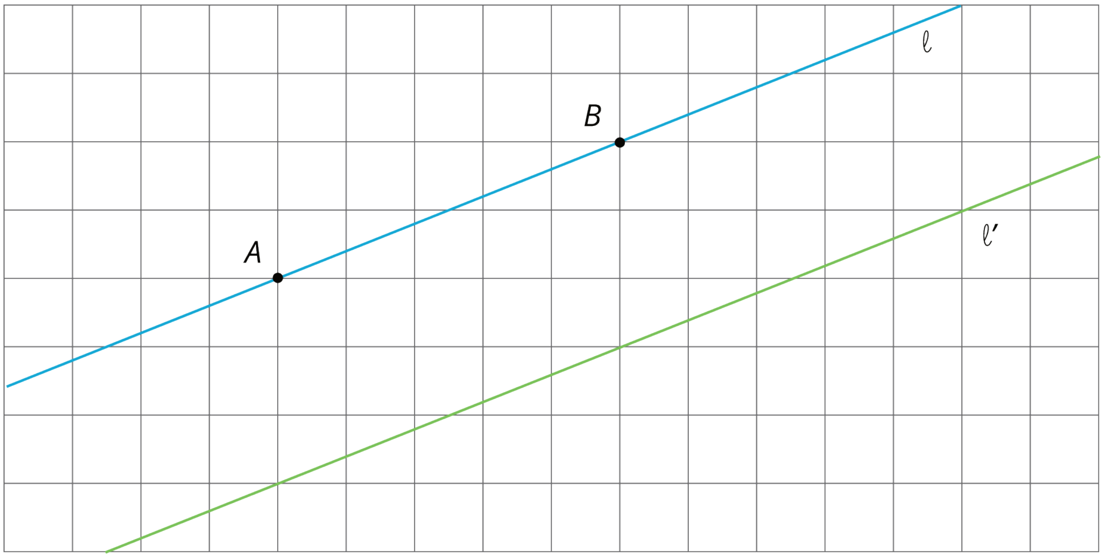
- 8.G.A.1.a

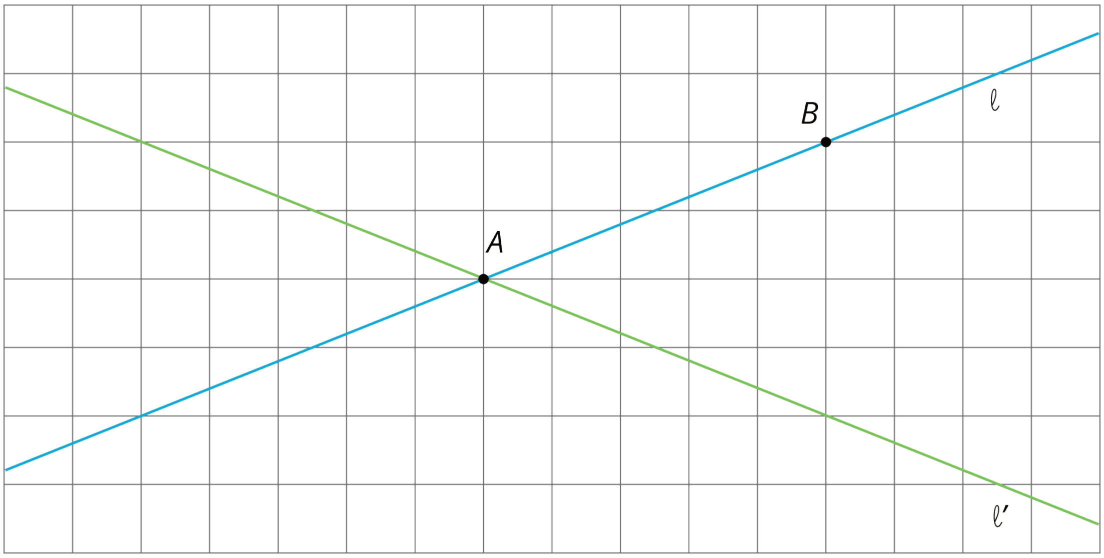
Launch

Provide access to tracing paper. Give students 2 minutes of quiet work time followed by whole-class discussion.

Student Task Statement

For each diagram, describe a translation, rotation, or reflection that takes line ℓ to line ℓ'. Then plot and label A' and B', the images of A and B.

1. 

2. 

Student Response

1. Answers vary. Possible responses:
    - Translation in many possible directions, for example, down 3 units
    - Reflection over a line parallel to ℓ halfway between ℓ and ℓ'
    - Rotation using a point halfway between ℓ and ℓ' as the center of rotation and an angle of 180°
2. Answers vary. Possible responses:
    - Reflection across the vertical line through point A

- Reflection across the horizontal line through point A
- Counterclockwise rotation about point A by the obtuse angle whose vertex is at A
- Clockwise rotation about point A by the acute angle whose vertex is at A

Activity Synthesis

Invite students to share the transformations they choose for each problem. Each diagram has more than one possible transformation that would result in the final figure. If the class only found one, pause for 2–3 minutes and encourage students to see if they can find another. For the first diagram, look for a single translation, single rotation, and single reflection that work. For the second diagram, look for a single rotation and a single reflection.

- "Will a translation work for the second diagram? Explain your reasoning." (A translation will not work. Since translations do not incorporate a turn, translations of a line are parallel to the original line or are the same line.)

9.2 Parallel Lines

15 minutes

In this activity, students will investigate the question, “What happens to parallel lines under rigid transformations?” by performing three different transformations on a set of parallel lines. After applying each transformation, they will jot down what they notice by answering the questions for each listed transformation.

As students work through these problems they may remember essential features of parallel lines (they do not meet, they remain the same distance apart). Rigid transformations do not change either of these features which means that the image of a set of parallel lines after a rigid transformation is another set of parallel lines (MP7).

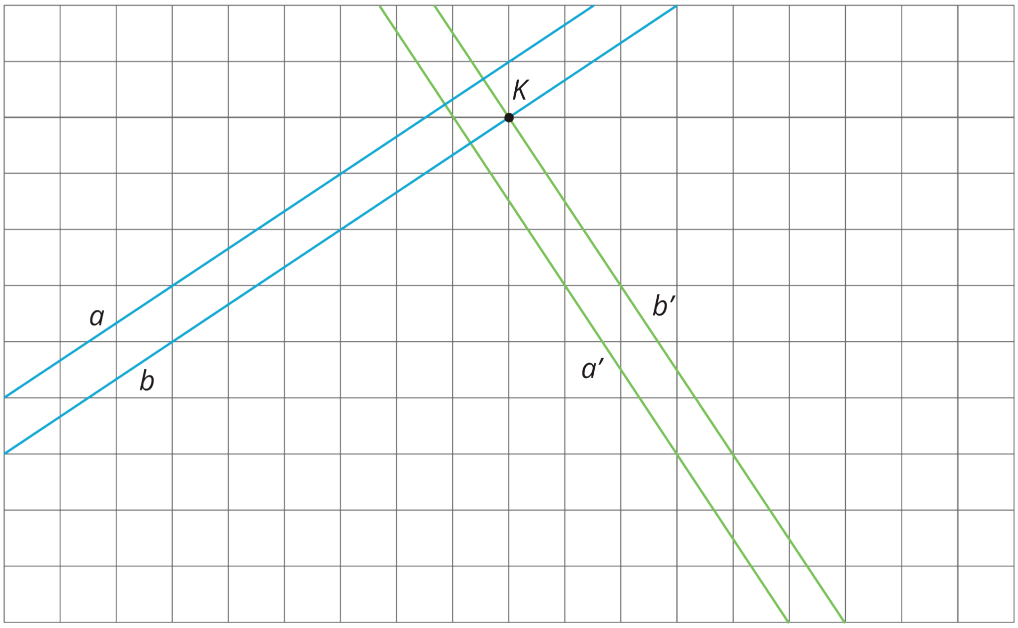
Identify the students who saw that the orientation of the lines changes but the lines remain parallel to each other regardless and select them to share during the discussion.

Addressing

- 8.G.A.1.c

Launch

Before beginning, review with students what happens when we perform a rigid transformation. Demonstrate by moving the tracing paper on top of the image to replicate an example transformation (for example, rotation of the lines clockwise 90° around the center K). Tell students that the purpose of this activity is to investigate, “What happens to parallel lines when we perform rigid transformations on them?”



Arrange students in groups of 3. Provide access to tracing paper. Each student in the group does *one* of the problems and then the group discusses their findings.

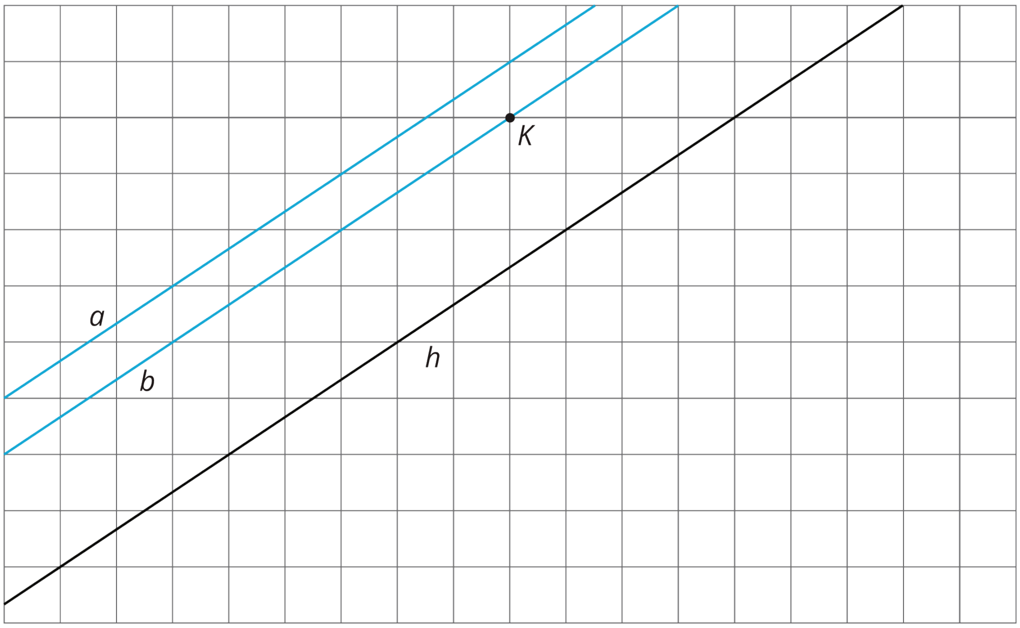
Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organizational skills in problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organization; Attention

Anticipated Misconceptions

Students may not perform the transformations on top of the original image. Ask these students to place the traced lines over the original and perform each transformation from there.

Student Task Statement



Use a piece of tracing paper to trace lines a and b and point K. Then use that tracing paper to draw the images of the lines under the three different transformations listed.

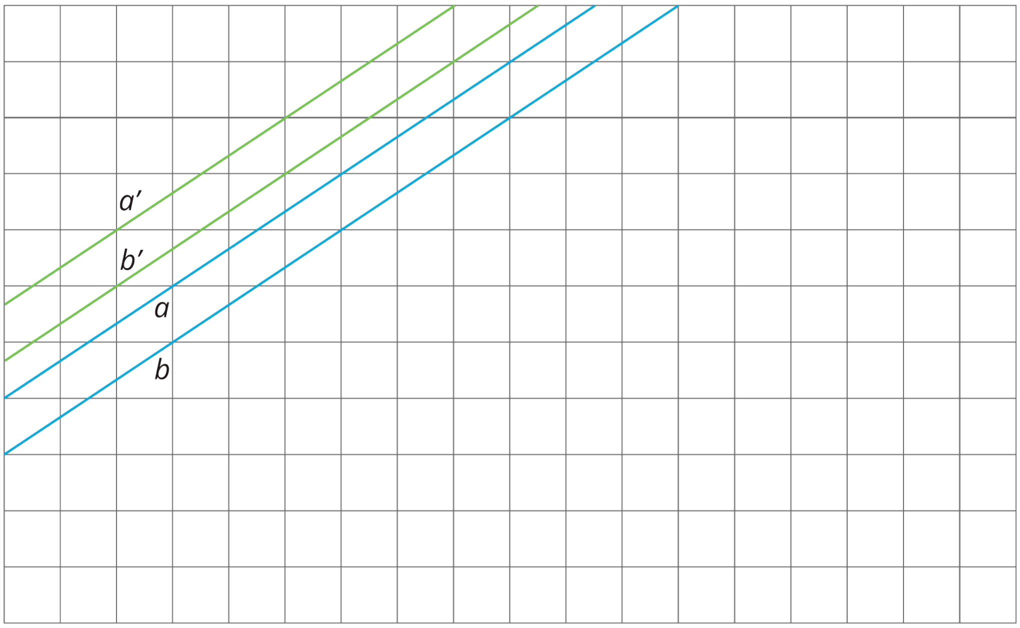
As you perform each transformation, think about the question:

What is the image of two parallel lines under a rigid transformation?

1. Translate lines a and b 3 units up and 2 units to the right.
    a. What do you notice about the changes that occur to lines a and b after the translation?
    b. What is the same in the original and the image?
2. Rotate lines a and b counterclockwise 180 degrees using K as the center of rotation.
    a. What do you notice about the changes that occur to lines a and b after the rotation?
    b. What is the same in the original and the image?
3. Reflect lines a and b across line h.
    a. What do you notice about the changes that occur to lines a and b after the reflection?
    b. What is the same in the original and the image?

Student Response

1. Translation: 3 grid square units up and 2 grid square units to the right

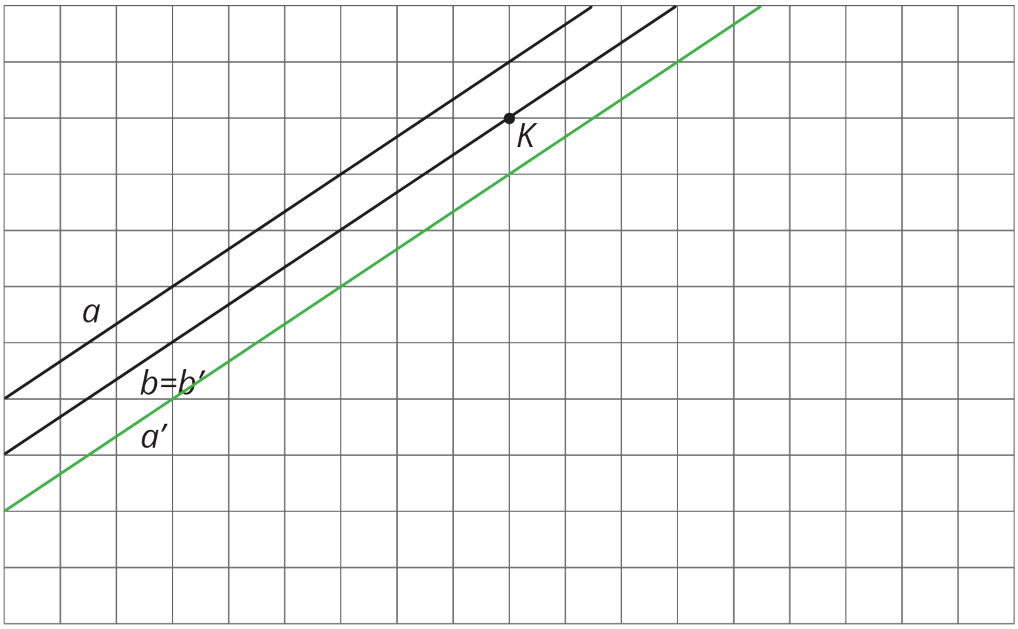


Answers vary. Sample Responses:

a. All 4 lines, a, b, a' , and b' are parallel. The lines a' and b' look like a and b but shifted upward.

b. The pair of lines remain parallel. The distance between the lines did not change.

2. Rotation around K

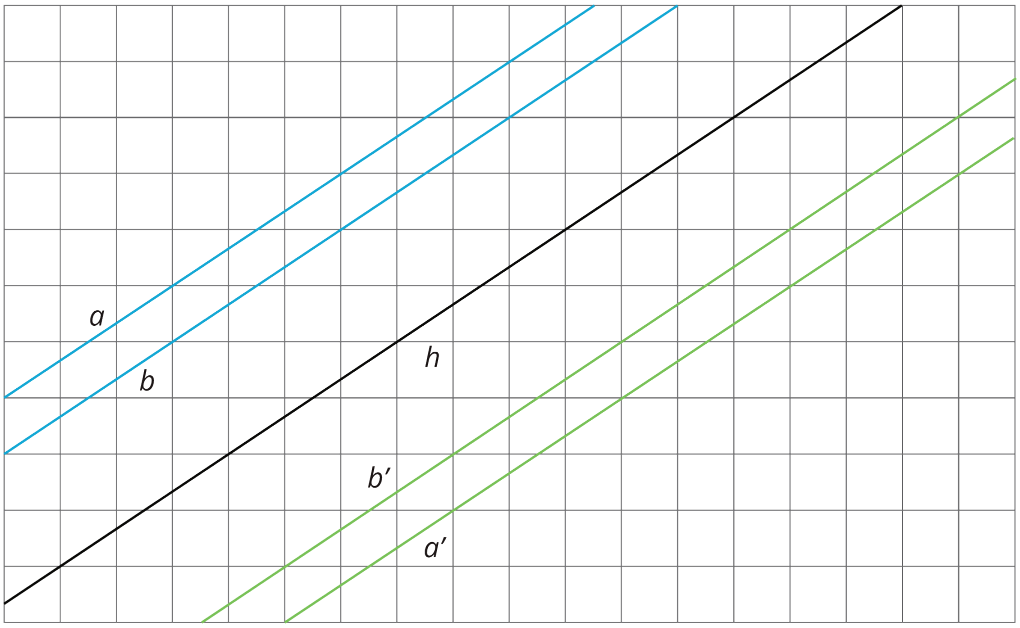


Answers vary. Sample responses:

a. The new pair of lines a' and b' are parallel to the original lines a and b.

b. The lines a' and b' are still parallel and they are the same distance apart as a and b.

3. Reflection over line h.



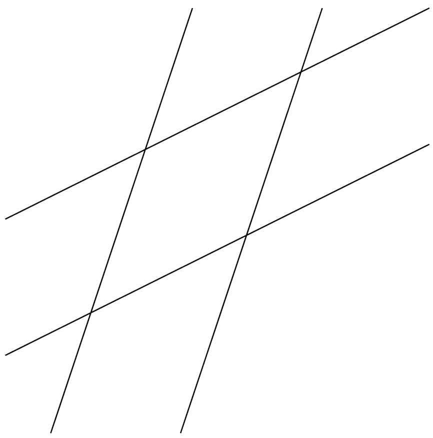
Answers vary. Sample responses:

a. Line a is above line b whereas line b' is above line a'.

b. Lines a' and b' are still parallel and are the same distance apart as lines a and b. All four lines are parallel to one another.

Are You Ready for More?

When you rotate two parallel lines, sometimes the two original lines intersect their images and form a quadrilateral. What is the most specific thing you can say about this quadrilateral? Can it be a square? A rhombus? A rectangle that isn't a square? Explain your reasoning.



Student Response

The quadrilateral is always a rhombus. It can be a square if the two pair of parallel lines are perpendicular. It can not be a rectangle that is not a square because the distance between the two sets of parallel lines is the same.

Activity Synthesis

Ask previously selected students who saw that the images of the parallel lines were parallel to the original in all three cases to share how they would answer the main question "What is the image of two parallel lines under a rigid transformation?" Make sure students understand that in general if ℓ and m are parallel lines and ℓ' and m' are their images under a rigid transformation then:

- ℓ' and m' are parallel.
- ℓ and m are *not* necessarily parallel to ℓ' and m' (refer to the 90 degree rotation shown during the launch).

In addition to the fact that the parallel lines remain parallel to each other when rigid transformations are performed, the distance between the lines stay the same. What can change is the position of the lines in the plane, in relative terms (i.e., which line is 'on top') or in absolute terms (i.e., does a line contain a particular point in the plane).

Give students 1–2 minutes of quiet time to write a response to the main question, "What is the image of two parallel lines under a rigid transformation?"

9.3 Let's Do Some 180's

15 minutes

In this activity, students apply their understanding of the properties of rigid transformations to 180° rotations of a line about a point on the line in order to establish the vertical angle theorem. Students have likely already used this theorem in grade 7, but this lesson informally demonstrates why the theorem is true. The demonstration of the vertical angle theorem exploits the structure of parallel lines and properties of both 180 degree rotations (studied in the previous lesson) and rigid transformations. This lesson is a good example of MP7, investigating the structure of different mathematical objects.

Students begin the activity by rotating a line with marked points 180° about a point on the line. Unlike the whole-class example discussed in the launch, this line contains marked points other than the center of rotation. Then students rotate an angle 180° about a point on the line to draw conclusions about lengths and angles. Finally, students are asked to consider the intersection of two lines, the angles formed, and how the measurements of those angles can be deduced using a 180° rotation about the intersection of the lines, which is the vertical angle theorem.

While students are working, encourage the use of tracing paper to show the transformations directly over the original image in order to help students keep track of what is happening with lines in each 180° rotation.

Building On

- 7.G.B.5

Addressing

- 8.G.A.1.a
- 8.G.A.1.b

Instructional Routines

- MLR7: Compare and Connect

Launch

Rotations require the students to think about rotating an entire figure. It would be good to remind students about this before the start of this activity. This might help students see what is happening in the first question better.

Before students read the activity, draw a line ℓ with a marked point D for all to see. Ask students to picture what the figure rotated 180° around point D looks like. After a minute of quiet think time, invite students share what they think the transformed figure would look like. Make sure all students agree that ℓ' looks "the same" as the original. If no students bring it up in their explanations, ask for suggestions of features that would make it possible to quickly tell the difference between the ℓ' and ℓ, such as another point or if the line were different colors on each side of point D.

Provide access to tracing paper.

Access for Students with Disabilities

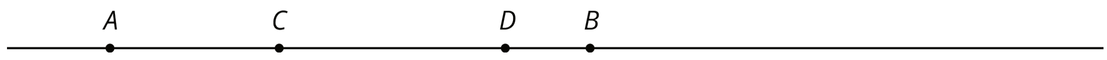
Engagement: Internalize Self Regulation. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. For example, to get students started, display only the first problem and diagram. Once students have successfully completed the three parts, introduce the remaining problems, one at a time. Invite students to think aloud about the relationships between the angles after the 180 degree rotation.
Supports accessibility for: Organization; Attention

Anticipated Misconceptions

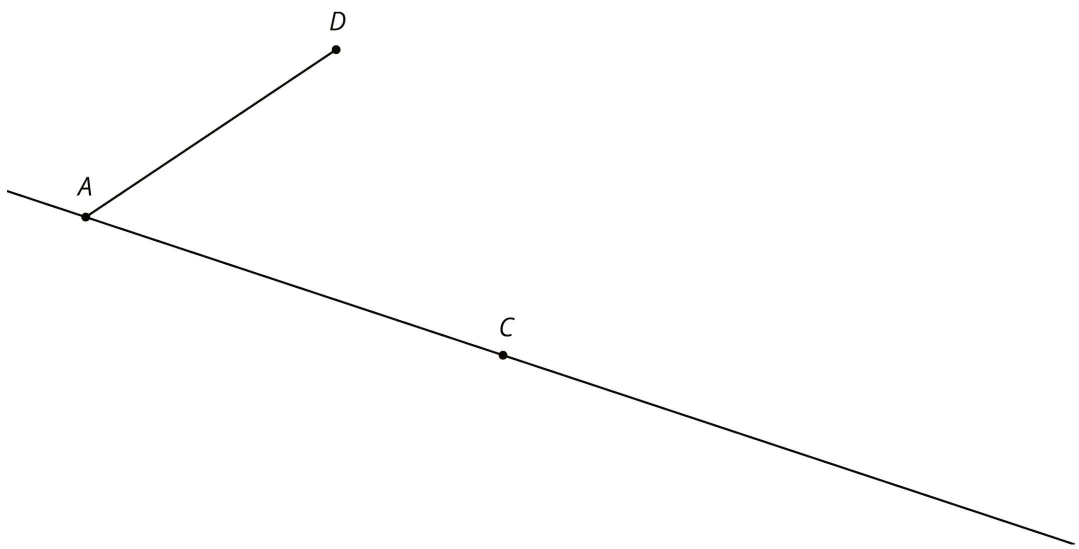
In the second question, students may not understand that rotating the figure includes both segment CA and segment AD since they have been working with rotating one segment at a time. Explain to these students that the figure refers to both of the segments. Encourage them to use tracing paper to help them visualize the rotation.

Student Task Statement

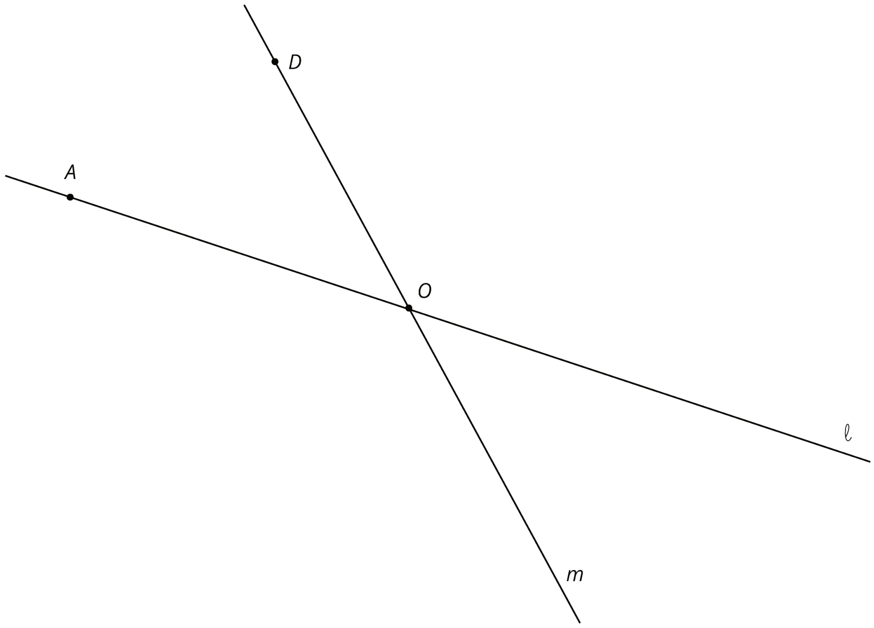
1. The diagram shows a line with points labeled A, C, D, and B.
   a. On the diagram, draw the image of the line and points A, C, and B after the line has been rotated 180 degrees around point D.
   b. Label the images of the points A', B', and C'.
   c. What is the order of all seven points? Explain or show your reasoning.



2. The diagram shows a line with points A and C on the line and a segment AD where D is not on the line.
   a. Rotate the figure 180 degrees about point C. Label the image of A as A' and the image of D as D'.
   b. What do you know about the relationship between angle CAD and angle $CA'D'$? Explain or show your reasoning.



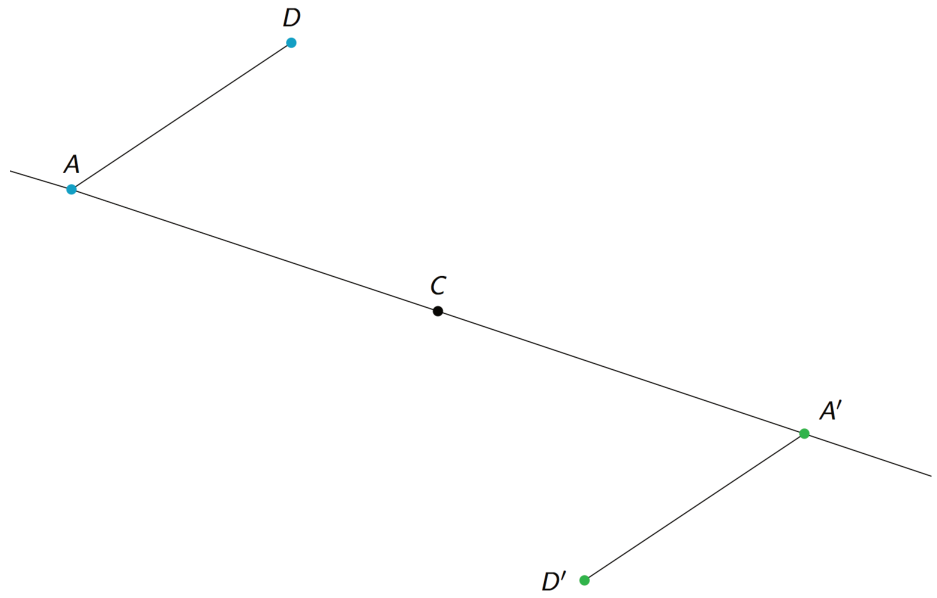
3. The diagram shows two lines ℓ and m that intersect at a point O with point A on ℓ and point D on m.
   a. Rotate the figure 180 degrees around O. Label the image of A as A' and the image of D as D'.
   b. What do you know about the relationship between the angles in the figure? Explain or show your reasoning.



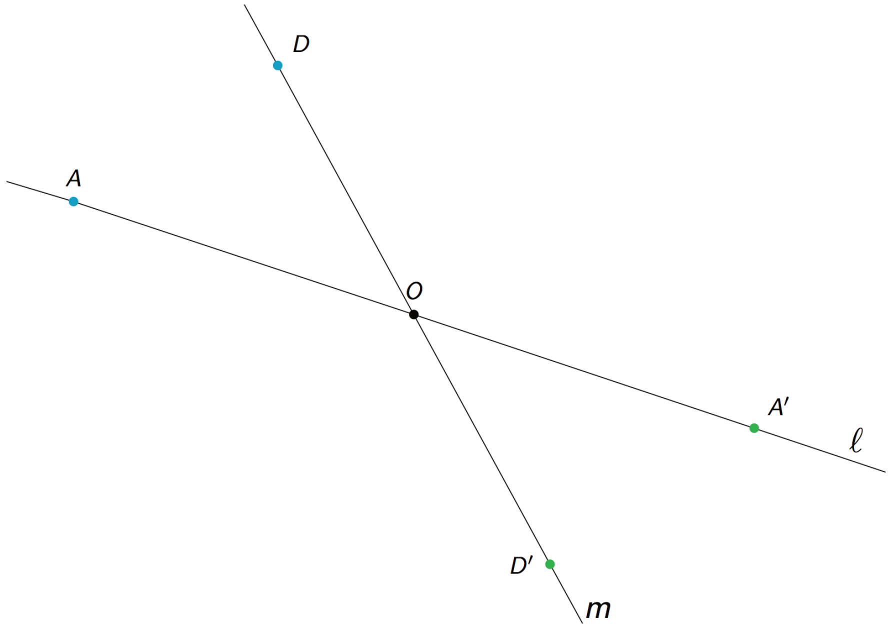
Student Response

1. A, C, B', D, B, C', A'

2. Lengths of segment CA and segment CA' are the same, lengths of segment AD and segment $A'D'$ are the same, and angles CAD and $CA'D'$ have the same measure because both distances and angle measures are preserved under rigid transformations.



3. Possible responses: Angles AOD and $A'OD'$ have the same measure. Angles DOA' and $D'OA$ have the same measure.



Activity Synthesis

The focus of the discussion should start with the relationships students find between the lengths of segments and angle measures and then move to the final problem, which establishes the vertical angle theorem as understood through rigid transformations. Questions to connect the discussion include:

- "What relationships between lengths did we find after performing transformations?" (They are the same.)

- "What relationships between angle measures did we find after performing transformations?" (They are the same.)
- "What does this transformation informally prove?" (Vertical angles are congruent.)

If time permits, consider discussing how the vertical angle theorem was approached in grade 7, namely by looking for pairs of supplementary angles. Pairs of vertical angles have the same measure because they are both supplementary to the *same* angle. The argument using 180 degree rotations is different because no reference needs to be made to the supplementary angle. The 180 degree rotation shows that both pairs of vertical angles have the same measure directly by mapping them to each other!

Access for English Language Learners

Representing, Speaking: MLR7 Compare and Connect. Use this routine when students share what they noticed about the relationships between the angle measures. Ask students to consider what changes and what stays the same when rigid transformations are applied to lines and segments. Draw students' attention to the associations between the rigid transformation, lengths of segments, and angle measures. These exchanges strengthen students' mathematical language use and reasoning based on rigid transformations of lines and will lead to the informal argument of the vertical angle theorem.
Design Principle(s): Maximize meta-awareness

Lesson Synthesis

In this lesson, students apply different rigid transformations to lines with a focus on parallel lines. They should be able to articulate what happens to parallel lines when a rigid transformation is performed on them. In addition, students gain a better understanding of why the **vertical angle** theorem they learned in grade 7 is true.

To highlight how transformations affect parallel lines, ask students:

- "When we perform rigid transformations on parallel lines, what do we know about their image?"
- "Does the distance between the lines change?"

To help students make a connection to how rotations affect lines in the second activity, ask:

- "When we rotate a line 180° around a point on the line where does the line land?"
- "How does the rotation affect the angle measurements for a pair of intersecting lines?"
- "How does this help us prove the vertical angle theorem?"

Students should see that a rotation of two intersecting lines about the point of intersection by 180° moves each angle to the angle that is vertical to it. Since rotation is a rigid transformation, the vertical angles must have the same measure.

In general, rigid transformations help us see that when we transform lines it might change the orientation but the lines retain their original properties.

9.4 Finding Missing Measurements

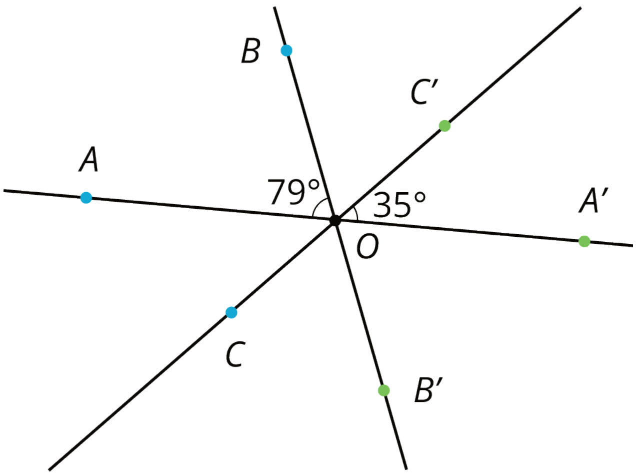
Cool Down: 5 minutes
Building directly from the previous activity, students fill in missing measurements using their understanding of both rigid transformations and the vertical angle theorem.

Addressing

- 8.G.A.1.a
- 8.G.A.1.b

Student Task Statement

Points A', B', and C' are the images of 180-degree rotations of A, B, and C, respectively, around point O.



Answer each question and explain your reasoning *without* measuring segments or angles.

1. Name a segment whose length is the same as segment AO.
2. What is the measure of angle $A'OB'$?

Student Response

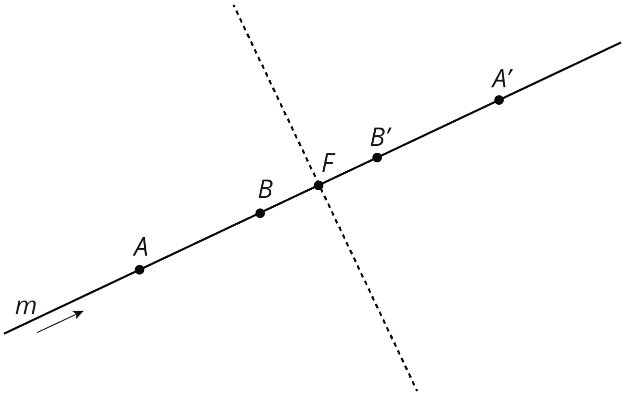
1. Segment $A'O$, because A' is the image of A after a 180 degree rotation with center at O. This rotation preserves distances and takes segment AO to segment $A'O$.

2. 79 degrees, the same measure as $\angle AOB$, because the 180 degree rotation with center at O takes $\angle AOB$ to $\angle A'OB'$. The rotation preserves angle measures.

Student Lesson Summary

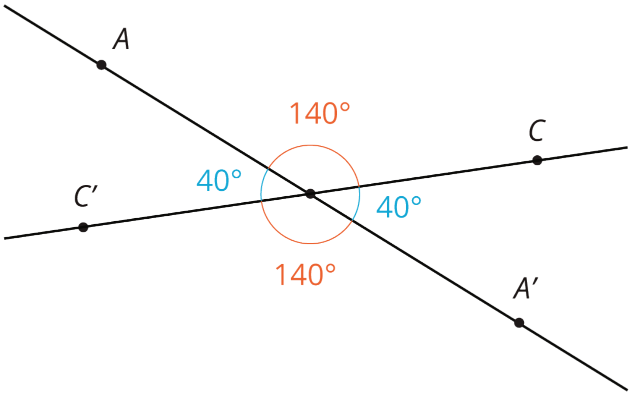
Rigid transformations have the following properties:

- A rigid transformation of a line is a line.
- A rigid transformation of two parallel lines results in two parallel lines that are the same distance apart as the original two lines.
- Sometimes, a rigid transformation takes a line to itself. For example:



  - A translation parallel to the line. The arrow shows a translation of line m that will take m to itself.
  - A rotation by 180° around any point on the line. A 180° rotation of line m around point F will take m to itself.
  - A reflection across any line perpendicular to the line. A reflection of line m across the dashed line will take m to itself.

These facts let us make an important conclusion. If two lines intersect at a point, which we'll call O, then a 180° rotation of the lines with center O shows that **vertical angles** are congruent. Here is an example:



Rotating both lines by 180° around O sends angle AOC to angle $A'OC'$, proving that they have the same measure. The rotation also sends angle AOC' to angle $A'OC$.

Glossary

- vertical angles

Lesson 9 Practice Problems

Problem 1

Statement

a. Draw parallel lines AB and CD.

b. Pick any point E. Rotate AB 90 degrees clockwise around E.

c. Rotate line CD 90 degrees clockwise around E.

d. What do you notice?

Solution

a. Answers vary.

b. Answers vary. The new line should be perpendicular to AB.

c. Answers vary. The new line should be perpendicular to CD and parallel to $A'B'$.

d. Answers vary. Sample response: the two new rotated lines are parallel.

Problem 2

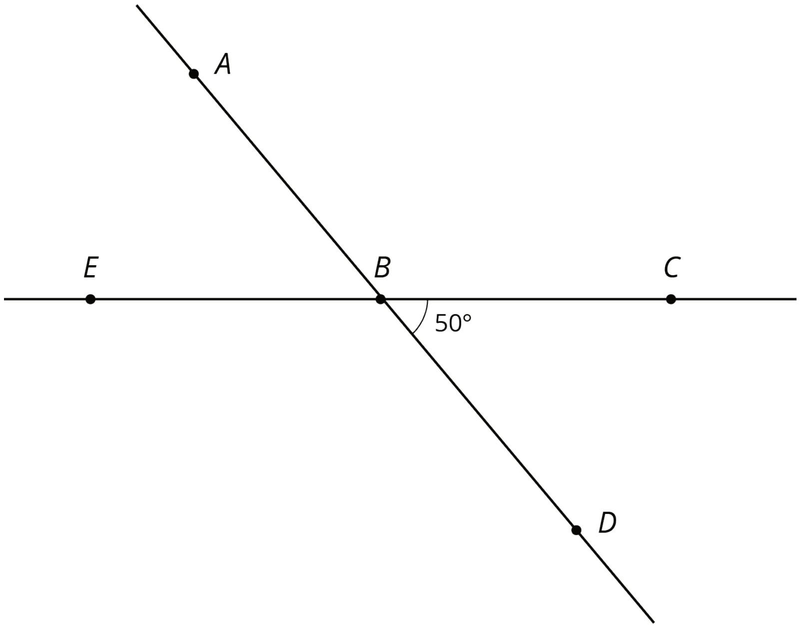
Statement

Use the diagram to find the measures of each angle. Explain your reasoning.

a. $m\angle ABC$

b. $m\angle EBD$

c. $m\angle ABE$



Solution

a. 130 degrees. $\angle ABC$ and $\angle CBD$ make a line, so they add up to 180 degrees.

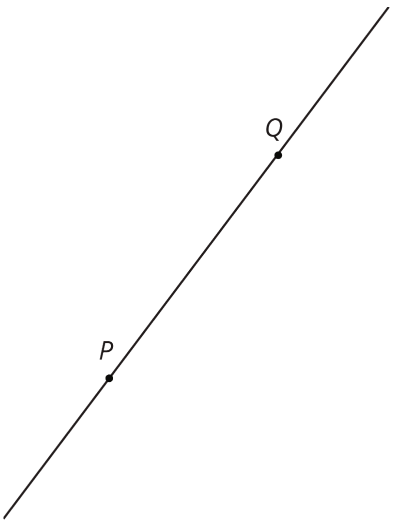
b. 130 degrees. $\angle EBD$ and $\angle CBD$ make a line, so they add up to 180 degrees.

c. 50 degrees. $\angle ABE$ and $\angle ABC$ make a line, so they add up to 180 degrees.

Problem 3

Statement

Points P and Q are plotted on a line.



a. Find a point R so that a 180-degree rotation with center R sends P to Q and Q to P.

b. Is there more than one point R that works for part a?

Solution

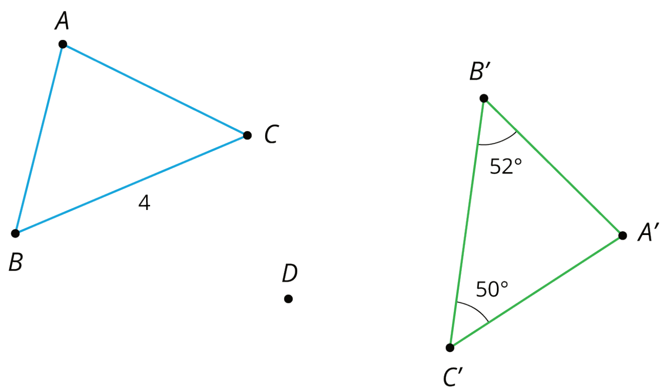
a. If R is the midpoint of segment PQ, then a rotation of 180 degrees with center R sends P to Q and Q to P.

b. No (The midpoint of PQ is the only point that works. 180-degree rotations with any other center do not send P to Q or Q to P.)

Problem 4

Statement

In the picture triangle $A'B'C'$ is an image of triangle ABC after a rotation. The center of rotation is D.



a. What is the length of side $B'C'$? Explain how you know.

b. What is the measure of angle B? Explain how you know.

c. What is the measure of angle C? Explain how you know.

Solution

a. 4 units. Rotations preserve side lengths, and side $B'C'$ corresponds to side BC under this rotation.

b. 52 degrees. Rotations preserve angle measures, and angles B and B' correspond to each other under this rotation.

c. 50 degrees. Rotations preserve angle measures, and angles C and C' correspond to each other under this rotation.

(From Unit 1, Lesson 7.)

Problem 5

Statement

The point $(-4, 1)$ is rotated 180 degrees counterclockwise using center $(0, 0)$. What are the coordinates of the image?

A. $(-1, -4)$

B. $(-1, 4)$

C. $(4, 1)$

D. $(4, -1)$

Solution

D

(From Unit 1, Lesson 6.)