Lesson 1: Tape Diagrams and Equations

Goals

- Draw tape diagrams to represent equations of the forms $x + p = q$ and $px = q$.
- Interpret (orally and in writing) tape diagrams that represent equations of the form $p + x = q$ or $px = q$.
- Use tape diagrams to find unknown values in equations of the forms $x + p = q$ and $px = q$ and explain (orally) the solution method.

Learning Targets

- I can tell whether or not an equation could represent a tape diagram.
- I can use a tape diagram to represent a situation.

Lesson Narrative

The purpose of this lesson is to help students remember from earlier grades how tape diagrams can be used to represent operations. There are two roles that tape diagrams (or any diagrams) can play: helping to visualize a relationship, and helping to solve a problem. The focus here is the first of these, so that later students can use diagrams for the second of these. In this lesson, students both interpret tape diagrams and create their own.

Note that the terms “solution” and “variable” aren’t defined until the next lesson, nor should any solution methods be generalized yet. Students should engage with the activities and reason about unknown quantities in ways that make sense to them.

Alignments

Addressing

- 6.EE.B.6: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

Building Towards

- 6.EE.B.5: Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.
- 6.EE.B.6: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

- 6.EE.B.7: Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR2: Collect and Display
- MLR7: Compare and Connect
- Think Pair Share

Student Learning Goals

Let's see how tape diagrams and equations can show relationships between amounts.

1.1 Which Diagram is Which?

Warm Up: 5 minutes
Students recall tape diagram representations of addition and multiplication relationships.

For relationships involving multiplication, we follow the convention that the first factor is the number of groups and the second is the number in each group. But students do not have to follow that convention; they may use their understanding of the commutative property of multiplication to represent relationships in ways that make sense to them.

Building Towards

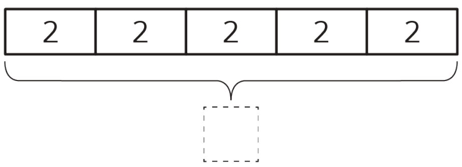
- 6.EE.B.6

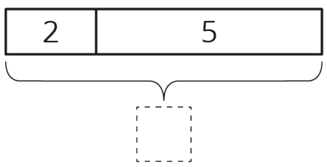
Launch

Give students 2 minutes of quiet think time, followed by a whole-class discussion.

Student Task Statement

1. Here are two diagrams. One represents $2 + 5 = 7$. The other represents $5 \cdot 2 = 10$. Which is which? Label the length of each diagram.





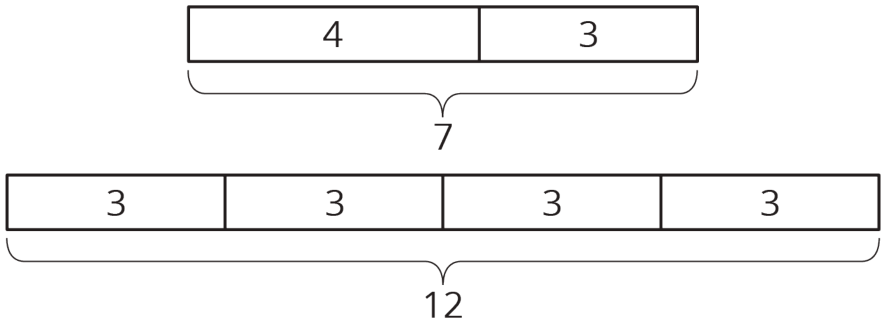
2. Draw a diagram that represents each equation.

$4 + 3 = 7$　　　　$4 \cdot 3 = 12$

Student Response

1. $5 \cdot 2 = 10$ on the left and $2 + 5 = 7$ on the right.

2.



Activity Synthesis

Invite students to share their responses and rationale. The purpose of the discussion is to give students an opportunity to articulate how operations can be represented by tape diagrams. Some questions to guide the discussion:

- "Where do you see the 5 in the first diagram?" (There are 5 equal parts represented by 5 same-size boxes.)
- "How did you find the length of the first diagram?" (Either computed $2+2+2+2+2$ or $5 \cdot 2$.)
- "Explain how you knew what the diagrams for $4+3=7$ and $4 \cdot 3=12$ should look like. How are they alike? How are they different?"
- "How did you represent $4 \cdot 3$? How are they alike? How are they different?" (Some may represent $4 \cdot 3$ as 4 groups of size 3, while some may represent as 3 groups of size 4.)

1.2 Match Equations and Tape Diagrams

10 minutes

In this first activity on tape diagram representations of equations with variables, students use what they know about relationships between operations to identify multiple equations that match a given diagram. It is assumed that students have seen representations like these in prior grades. If this is not the case, return to the examples in the warm-up and ask students to write other equations for each of the diagrams. For example, the relationship between the quantities 2, 5, and 7 expressed by the equation $2+5=7$ can also be written as $2=7-5$, $5=7-2$, $7=2+5$, and $7-2=5$. Ask students to explain how these equations match the parts of the tape diagram.

Note that the word "variable" is not defined until the next lesson. It is not necessary to use that term with students yet. Also, we are sneaking in the equivalent expressions $x+x+x+x$ and $4 \cdot x$ because these equivalent ways of writing it should be familiar from earlier grades, but *equivalent expressions* are defined more carefully later in this unit. Even though this familiar example appears, the general idea of equivalent expressions is not a focus of this lesson.

Building Towards

- 6.EE.B.6
- 6.EE.B.7

Instructional Routines

- MLR2: Collect and Display
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 2 minutes of quiet work time. Then, ask them to share their responses with their partner and follow with whole-class discussion.

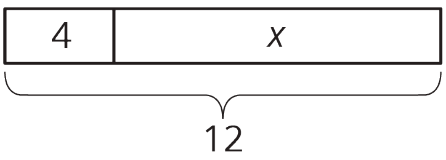
If necessary, explain that the x in each diagram is just standing in for a number.

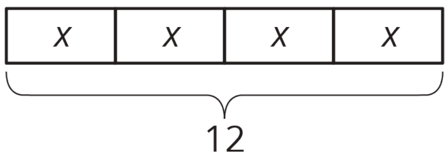
Anticipated Misconceptions

Students may not have much experience with a letter standing in for a number. If students resist engaging, explain that the x is just standing in for a number. Students may prefer to figure out the value that x must take to make each diagram make sense (8 in the first diagram and 3 in the second diagram) before thinking out which equations can represent each diagram.

Student Task Statement

Here are two tape diagrams. Match each equation to one of the tape diagrams.





- $4 + x = 12$
- $12 \div 4 = x$
- $4 \cdot x = 12$
- $12 = 4 + x$
- $12 - x = 4$
- $12 = 4 \cdot x$
- $12 - 4 = x$
- $x = 12 - 4$
- $x + x + x + x = 12$

Student Response

Left diagram:

- $4 + x = 12$
- $12 = 4 + x$
- $12 - x = 4$
- $12 - 4 = x$
- $x = 12 - 4$

Right diagram:

- $12 \div 4 = x$
- $4 \cdot x = 12$
- $12 = 4 \cdot x$
- $x + x + x + x = 12$

Activity Synthesis

Focus the discussion on the reason that more than one equation can describe each tape diagram; namely, the same relationship can be expressed in more than one way. These ideas should be familiar to students from prior work. Ideas that may be noted:

- A multiplicative relationship can be expressed using division.
- An additive relationship can be expressed using subtraction.
- It does not matter how expressions are arranged around an equal sign. For example, $4 + x = 12$ and $12 = 4 + x$ mean the same thing.
- Repeated addition can be represented with multiplication. For example, $4x$ is another way to express $x + x + x + x$.

Students are likely to express these ideas using informal language, and that is okay. Encourage them to revise their statements using more precise language, but there is no reason to insist they use particular terms.

Some guiding questions:

- “How can you tell if a diagram represents addition or multiplication?”
- “Once you were sure about one equation, how did you find others that matched the same diagram?”
- Regarding any two equations that represent the same diagram: “What is the same about the equations? What is different?”

Access for English Language Learners

Writing, Representing, Conversing: MLR2 Collect and Display. While pairs are working, circulate and listen to student talk about the similarities and differences between the tape diagrams and the equations. Ask students to explain how these equations match the parts of the tape diagram. Write down common or important phrases you hear students say about each representation onto a visual display of both the tape diagrams and the equations. This will help the students use mathematical language during their paired and whole-group discussions.
Design Principle(s): Support sense-making; Cultivate conversation

1.3 Draw Diagrams for Equations

15 minutes

In this activity, students draw tape diagrams to match given equations. Then, they reason about the unknown value that makes the equation true, a process also known as solving the equation. Students should not be shown strategies to solve but rather should reason with equations or diagrams in ways that make sense to them. As they work, monitor for students who use the diagrams to find unknown quantities and for those who use the equations.

Building Towards

- 6.EE.B.5
- 6.EE.B.7

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR7: Compare and Connect

Launch

Give students 5 minutes quiet work time followed by a whole-class discussion.

Access for Students with Disabilities

Representation: Internalize Comprehension. For each equation, provide students with a blank template of a tape diagram for students to complete and find the unknown quantities.
Supports accessibility for: Visual-spatial processing; Organization

Anticipated Misconceptions

Students might draw a box with 3 for the equation $18 = 3 \cdot y$. Ask students about the meaning of multiplication and specifically what $3 \cdot y$ means. Ask how they could represent 3 equal groups with unknown size y.

Students might think they need to show an unknown number (y) of equal groups of 3. While this is possible, showing 3 equal groups with unknown size y is simpler to represent.

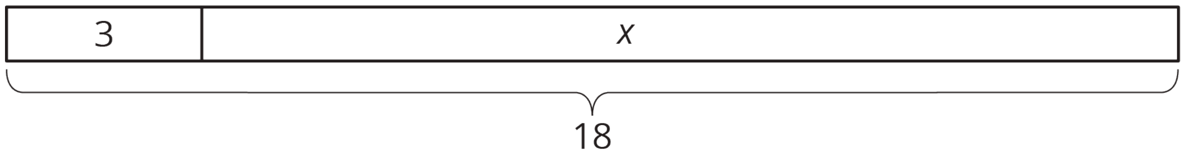
Student Task Statement

For each equation, draw a diagram and find the value of the unknown that makes the equation true.

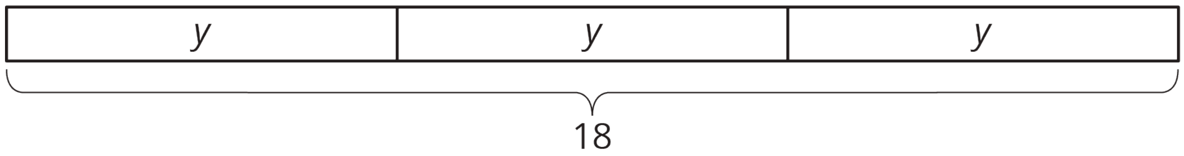
1. $18 = 3 + x$
2. $18 = 3 \cdot y$

Student Response

1. For $18 = 3 + x$, the value is 15.



2. For $18 = 3 \cdot y$, the value is 6.



Are You Ready for More?

You are walking down a road, seeking treasure. The road branches off into three paths. A guard stands in each path. You know that only one of the guards is telling the truth, and the other two are lying. Here is what they say:

- Guard 1: The treasure lies down this path.
- Guard 2: No treasure lies down this path; seek elsewhere.
- Guard 3: The first guard is lying.

Which path leads to the treasure?

Student Response

Path 2 leads to the treasure.

Suppose Guard 1 is telling the truth. Then it would be true that Path 1 leads to the treasure. Then Guard 2’s statement must be true as well. But only one of the guards is telling the truth. This means that Guard 1 must be lying.

Since Guard 1 is lying, Guard 3 is telling the truth about Guard 1 lying. That means that Guard 3 has the one true statement. Guard 2, then, is lying about his path being the wrong path, so the treasure lies down Path 2.

Activity Synthesis

Invite students to share their strategies for finding the values of x and y. Include at least one student who reasoned with the diagram and one who reasoned with the equation. Help students connect different methods by thinking about the relationships between the three quantities in each problem and how both the equations and the diagrams represent them.

Access for English Language Learners

Speaking, Listening, Representing: MLR7 Compare and Connect. Use this routine when students present their equations and tape diagrams. Draw students' attention to how the mathematical operations (addition, subtraction, multiplication, division) are represented in each relationship. For example, ask, "Where do you see multiplication in the diagram?" This will strengthen students' mathematical language use and reasoning about tape diagram representations of the equations.
Design Principle(s): Support sense-making; Maximize meta-awareness

Lesson Synthesis

To ensure that students understand the use and usefulness of tape diagrams in representing equations and finding unknown values, consider asking some of the following questions:

- "Why are tape diagrams useful to visualize a relationship?" (Answers vary. Sample response: You can see the way quantities are related.)
- "Where in the tape diagram do we see the equal sign that is in the equation it represents?" (The fact that the sum of the parts has the same value as the whole; the numbers and letters in the boxes add up to the total shown for the whole rectangle.)
- "Why can a diagram be represented by more than one equation?" (Because more than one operation can be used; for example, the same diagram can be represented by an addition or a subtraction equation. Because when two expressions are equal, it doesn't matter how they are arranged around the equal sign.)
- "Describe some ways to represent the relationship $23 + x = 132$" (Tape diagram with two unequal parts, other equivalent equations like $x = 132 - 23$).
- "Describe some ways to represent the relationship $5x = 230$" (Tape diagram with 5 equal parts, other equivalent equations like $x = 230 \div 5$).

1.4 Finish the Diagrams

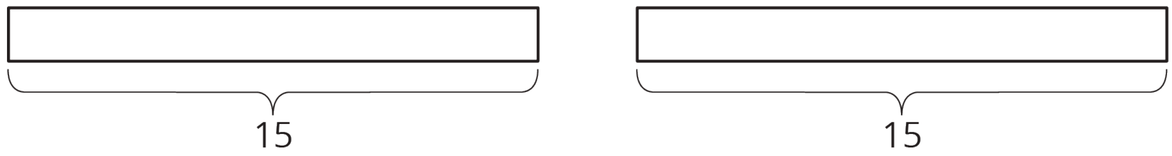
Cool Down: 5 minutes

Addressing

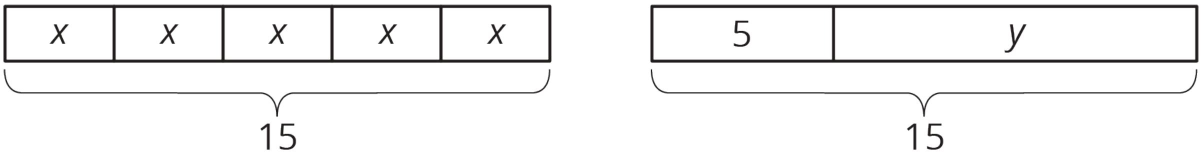
- 6.EE.B.6

Student Task Statement

Finish the first diagram so that it represents $5 \cdot x = 15$, and the second diagram so that it represents $5 + y = 15$.



Student Response



Student Lesson Summary

Tape diagrams can help us understand relationships between quantities and how operations describe those relationships.

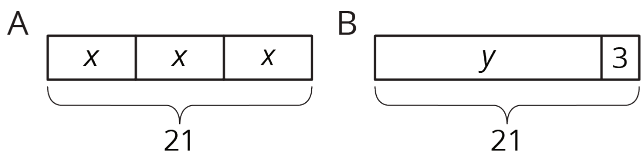


Diagram A has 3 parts that add to 21. Each part is labeled with the same letter, so we know the three parts are equal. Here are some equations that all represent diagram A:

$$x + x + x = 21$$

$$3 \cdot x = 21$$

$$x = 21 \div 3$$

$$x = \frac{1}{3} \cdot 21$$

Notice that the number 3 is not seen in the diagram; the 3 comes from counting 3 boxes representing 3 equal parts in 21.

We can use the diagram or any of the equations to reason that the value of x is 7.

Diagram B has 2 parts that add to 21. Here are some equations that all represent diagram B:

$$y + 3 = 21$$

$$y = 21 - 3$$

$$3 = 21 - y$$

We can use the diagram or any of the equations to reason that the value of y is 18.

Lesson 1 Practice Problems

Problem 1

Statement

Here is an equation: $x + 4 = 17$

a. Draw a tape diagram to represent the equation.

b. Which part of the diagram shows the quantity x? What about 4? What about 17?

c. How does the diagram show that $x + 4$ has the same value as 17?

Solution

a. A tape diagram showing one part labeled x and another labeled 4 and a total of 17.

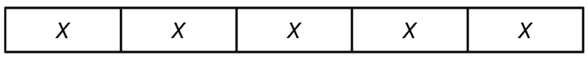
b. The rectangle labeled x represents the quantity x, and the rectangle labeled 4 represents the quantity 4. The big rectangle (the combination of the two smaller ones) represents 17.

c. The large rectangle is labeled 17, but it is also obtained by joining the two smaller rectangles labeled x and 4.

Problem 2

Statement

Diego is trying to find the value of x in $5 \cdot x = 35$. He draws this diagram but is not certain how to proceed.

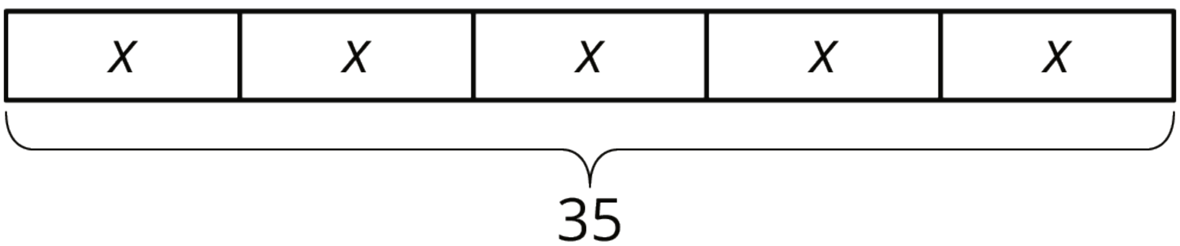


a. Complete the tape diagram so it represents the equation $5 \cdot x = 35$.

b. Find the value of x.

Solution

a.



b. $x = 7$

Problem 3

Statement

Match each equation to one of the two tape diagrams.

a. $x+3=9$

b. $3 \cdot x = 9$

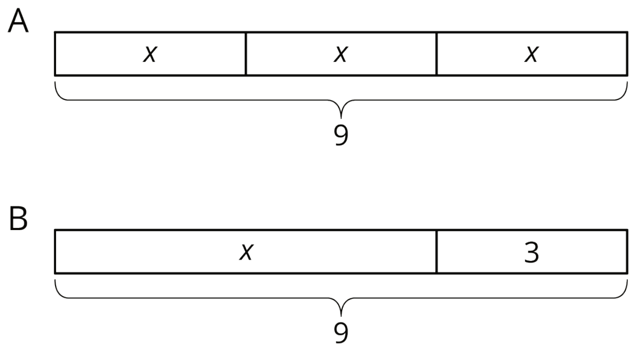
c. $9 = 3 \cdot x$

d. $3 + x = 9$

e. $x = 9 - 3$

f. $x = 9 \div 3$

g. $x + x + x = 9$



Solution

a. B

b. A

c. A

d. B

e. B

f. A

g. A

Problem 4

Statement

For each equation, draw a tape diagram and find the unknown value.

a. $x + 9 = 16$

b. $4 \cdot x = 28$

Solution

a. A tape diagram showing one part labeled 9 and another labeled x and a total of 16. The solution is 7.

b. A tape diagram showing 4 groups labeled x and a total of 28. The solution is 7.

Problem 5

Statement

A shopper paid \$2.52 for 4.5 pounds of potatoes, \$7.75 for 2.5 pounds of broccoli, and \$2.45 for 2.5 pounds of pears. What is the unit price of each item she bought? Show your reasoning.

Solution

Potatoes cost \$0.56 per pound, broccoli costs \$3.10 per pound, and pears costs \$0.98 per pound. Reasoning varies. Sample reasoning:

- $2.52 \div 4.5 = 252 \div 450$, which equals 0.56.
- $7.75 \div 2.5 = 775 \div 250$, which equals 3.1 or 3.10.
- $2.45 \div 2.5 = 245 \div 250$, which equals 0.98.

(From Unit 5, Lesson 13.)

Problem 6

Statement

A sports drink bottle contains 16.9 fluid ounces. Andre drank 80% of the bottle. How many fluid ounces did Andre drink? Show your reasoning.

Solution

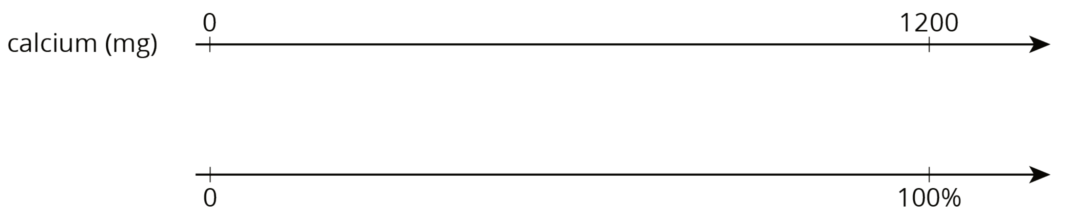
13.52 fluid ounces ($0.8 \cdot 16.9 = 13.52$)

(From Unit 3, Lesson 14.)

Problem 7

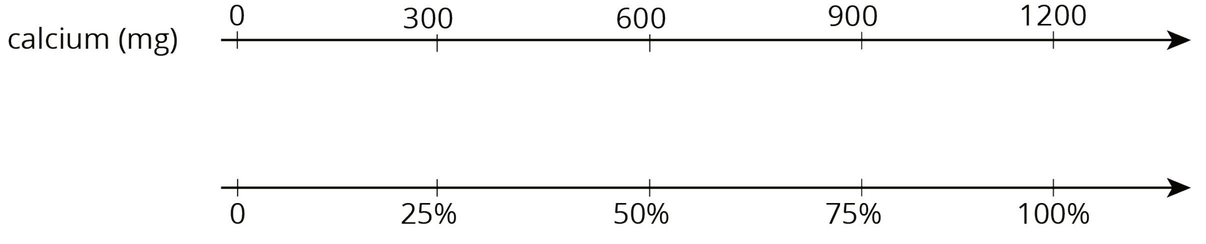
Statement

The daily recommended allowance of calcium for a sixth grader is 1,200 mg. One cup of milk has 25% of the recommended daily allowance of calcium. How many milligrams of calcium are in a cup of milk? If you get stuck, consider using the double number line.



Solution

300 mg. Sample reasoning using double number line:



(From Unit 3, Lesson 11.)