Lesson 8: Relating Area to Circumference

Goals

- Generalize a process for finding the area of a circle, and justify (orally) why this can be abstracted as πr^2.
- Show how a circle can be decomposed and rearranged to approximate a polygon, and justify (orally and in writing) that the area of this polygon is equal to half of the circle's circumference multiplied by its radius.

Learning Targets

- I can explain how the area of a circle and its circumference are related to each other.
- I know the formula for area of a circle.

Lesson Narrative

In the previous lesson, students found that it takes a little more than 3 squares with side lengths equal to the circle's radius to completely cover a circle. Students may have predicted that the area of a circle can be found by multiplying πr^2. In this lesson students derive that relationship through informal dissection arguments. In the main activity they cut and rearrange a circle into a shape that approximates a parallelogram (MP 3). In an optional activity, they consider a different way to cut and rearrange a circle into a shape that approximates a triangle. In both arguments, one side of the polygon comes from the circumference of the circle, leading to the presence of π in the formula for the area of a circle.

Alignments

Addressing

- 7.G.B.4: Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR2: Collect and Display
- Notice and Wonder
- Think Pair Share

Required Materials

Blank paper
Cylindrical household items
Glue or glue sticks
Markers
Scissors

Required Preparation

You will need one cylindrical household item (like a can of soup) for each group of 2 students. The activity works best if the diameter of the item is between 3 and 5 inches.

If possible, it would be best to give each group 2 different colors of blank paper.

Student Learning Goals

Let’s rearrange circles to calculate their areas.

8.1 Irrigating a Field

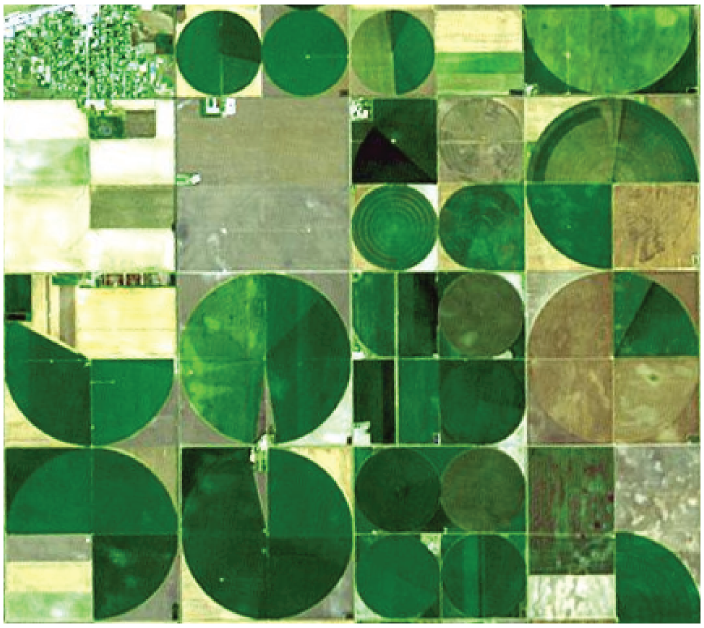
Warm Up: 5 minutes
The purpose of this activity is for students to estimate the area of a circle by comparing it to a surrounding square.

Addressing

- 7.G.B.4

Launch

Explain that some farms have circular fields because they use center-pivot irrigation. If desired, display these images to familiarize students with the context.



Provide quiet think time followed by whole-group discussion.

Anticipated Misconceptions

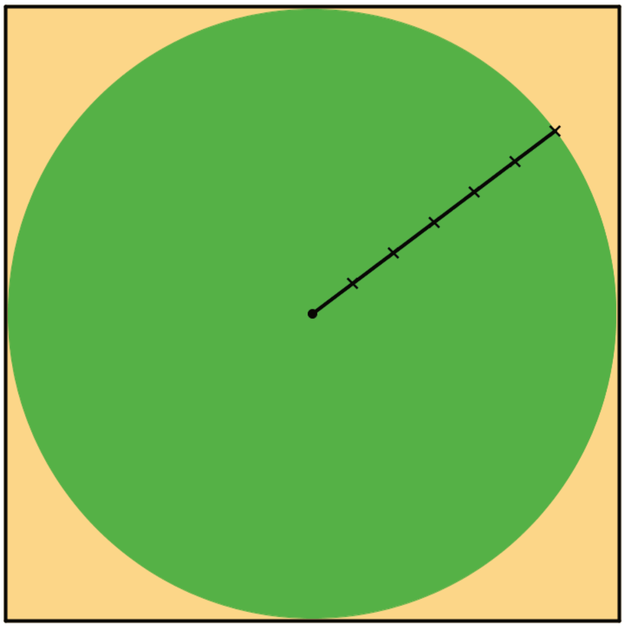
Students might think the answer should be 640,000 m^2 because that is the area of the square, not realizing that they are being asked to find the area of a circle. Ask them what shape is the region where the plants are growing.

Some students might incorrectly calculate the area of the square to be 6,400 m^2 and therefore estimate that the circle would be about 5,000 m^2.

Some students might try to use what they learned in the previous lessons about the relationship between the area of a circle and the area of a square with side length equal to the circle's radius. Point out that the question is asking for an estimate and answer choices all differ by a factor of 10.

Student Task Statement

A circular field is set into a square with an 800 m side length. Estimate the field's area.



- About 5,000 m^2
- About 50,000 m^2
- About 500,000 m^2
- About 5,000,000 m^2
- About 50,000,000 m^2

Student Response

C. The area of the circular field could be about 500,000 m^2 because it needs to be slightly less than the area of the square around it, which is 640,000 m^2, because $800 \cdot 800 = 640{,}000$.

Activity Synthesis

Discuss the estimation strategies students used to answer the question. Ask students what the area of the square is in square meters ($800 \cdot 800$, or 640,000). Ask them if the circle's area is greater than or less than the square's area (less). Then ask them to use the picture to determine the best estimate (500,000 since the circle is close in area to the square).

8.2 Making a Polygon out of a Circle

20 minutes

The purpose of this activity is for students to use what they know about finding the area of a parallelogram to develop the formula for the area of a circle. This activity builds on the work students did in grade 6 when cutting and rearranging shapes in order to calculate their areas. In this activity, students cut and rearrange parts of a circle to approximate a parallelogram. They see that the area of the parallelogram would be calculated by multiplying half of the circle's circumference times its radius. Since students are not familiar with the process of writing proofs, it is necessary to walk them through writing the justification that uses the formula for the area of the parallelogram to develop the formula for the area of the circle.

The construction in this activity shows that the constructed parallelogram has a height at most the radius of the circle and a base at most half the circumference of the circle. Establishing equality is beyond grade level and will be addressed again in high school.

The process used to decompose the circle and recompose it into a shape resembling a parallelogram is a good example of MP8. The pie shaped wedges are successively cut in half and rearranged. Each time, the sides of the shape look more like line segments.

Watch for students who identify that the rearranged circle pieces resemble a parallelogram as the pieces get smaller. Also watch for how they estimate the width and height of this parallelogram and invite them to share during the discussion.

Addressing

- 7.G.B.4

Instructional Routines

- MLR2: Collect and Display

Launch

Arrange students in groups of 2. Each group needs a circular object, with a diameter between 3 and 5 inches, and a thick marker with which to trace it. Also provide each group a sheet of white paper, a sheet of colored paper, a pair of scissors, and glue or tape. Remind students that in the past they decomposed and rearranged a shape to figure out its area. Demonstrate how to do the first 4 steps of the activity, and invite students to follow along with your example. Ask how the area of the new shape differs from that of the circle. Solicit some ideas on what the new shape resembles and how the area of such a shape could be approximated. Without resolving this, ask students to continue the process.

Access for Students with Disabilities

Engagement: Develop Effort and Persistence. Provide prompts, reminders, guides, rubrics, or checklists that focus on increasing the length of on-task orientation in the face of distractions. For example, provide students with a task checklist which makes all the required components of the visual display explicit.
Supports accessibility for: Attention; Social-emotional skills

Access for English Language Learners

Conversing, Reading: MLR2 Collect and Display. As students work in groups to make sense of the shapes glued on the paper, circulate and listen to the language students use as they compare the shapes and discuss how to find the area of the shape that resembles a parallelogram. Write down the words and phrases students use to explain why the areas of both shapes are equal and why the area of the shape is half of the circumference multiplied by the radius. As students review the language collected in the visual display, encourage students to clarify the meaning of a word or phrase. For example, a phrase such as "the base is half of the circle" can be clarified with the phrase, "the base of the parallelogram is half of the circumference of the circle." A phrase such as, "the height is the radius" can be clarified with the phrase, "the height of the parallelogram is equal to the radius of the circle." This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-awareness of language.
Design Principle(s): Support sense-making; Maximize meta-awareness

Anticipated Misconceptions

Students might not fold the wedges accurately or make a straight cut. Remind them that the halves must be equal.

Student Task Statement

Your teacher will give you a circular object, a marker, and two pieces of paper of different colors.

Follow these instructions to create a visual display:

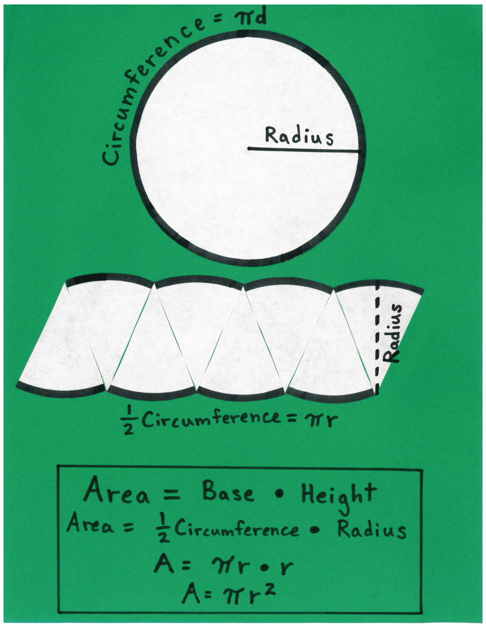
1. Using a thick marker, trace your circle in two separate places on the same piece of paper.
2. Cut out both circles, cutting around the marker line.
3. Fold and cut one of the circles into fourths.
4. Arrange the fourths so that straight sides are next to each other, but the curved edges are alternately on top and on bottom. Pause here so your teacher can review your work.
5. Fold and cut the fourths in half to make eighths. Arrange the eighths next to each other, like you did with the fourths.
6. If your pieces are still large enough, repeat the previous step to make sixteenths.
7. Glue the remaining circle and the new shape onto a piece of paper that is a different color.

After you finish gluing your shapes, answer the following questions.

1. How do the areas of the two shapes compare?
2. What polygon does the shape made of the circle pieces most resemble?
3. How could you find the area of this polygon?

Student Response

1. - 7. The shapes shown here, but without anything labeled.

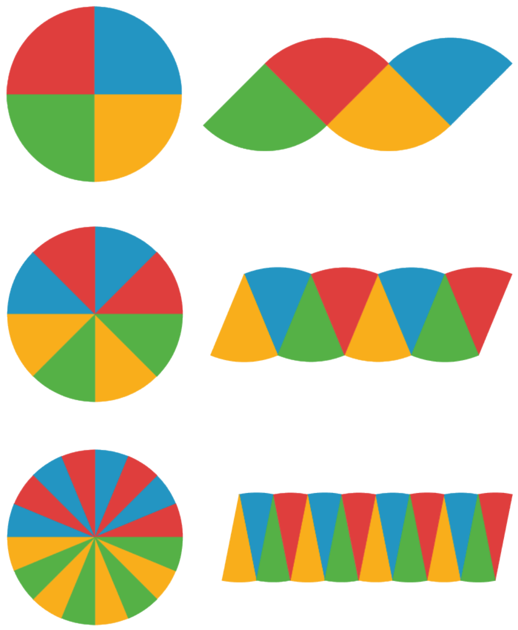


1. They are equal.
2. It is like a parallelogram but with a squiggly top and bottom.
3. Multiply base times height.

Activity Synthesis

Ask students which polygon resembles the shape they constructed of circle pieces and how they know. (A parallelogram or rectangle)

For a dynamic visualization, see http://ggbm.at/RUqSMrjn, created in GeoGebra by Malin Christersson, or display this image.



Ask, “If we could continue cutting the wedges in half, how would that affect the new shape?” (It would look even more like a parallelogram or rectangle. The bumpy top and bottom straighten out, and the slanted height becomes more vertical.)

Note that we are going to refer to the bumpy parallelogram-ish shape as the “parallelogram” (in quotes). Ask students to describe comparisons we can make between the measurements in the circle and the “parallelogram.” Record and display these ideas for all to see. It may be helpful to write over the actual images themselves. Students should notice the following measurements, but if they do not, prompt them to look for them:

- The base of the “parallelogram” is approximately equal to half of the circle’s circumference.
- The height of the “parallelogram” is approximately equal to the radius of the circle.
- The areas of the 2 shapes are equal.

Tell students to label these measurements on their visual display:

- “$\text{Circumference} = \pi d$” around the circle
- “$\frac{1}{2}\text{Circumference} = \pi r$” at the base of the “parallelogram”
- “Radius” on the radius of the circle (needs to be drawn in)
- “Radius” on the height of the “parallelogram” (needs to be drawn in)

If students struggle to understand these relationships through the abstract variables, consider measuring your example circle and using the numerical measurements to talk about these relationships.

Ask students to discuss the different ways we can calculate the area of the "parallelogram." Students should share the following ways:

- $\text{Area} = \text{Base} \boldsymbol{\cdot} \text{Height}$
- $\text{Area} = \frac{1}{2}\text{Circumference} \boldsymbol{\cdot} \text{Radius}$
- $\text{A} = \pi r \boldsymbol{\cdot} r$
- $\text{A} = \pi r^2$

Tell students to record these ideas on their visual display. Display student work for all to see. If students do not bring up one of these ideas, make it explicit in the discussion.

8.3 Making Another Polygon out of a Circle

Optional: 10 minutes (there is a digital version of this activity)
The purpose of this activity is for students to consider a different way to cut and reassemble a circle into something resembling a polygon in order to calculate its area. This time the polygon is a triangle, but the area of the circle can still be found by multiplying $\frac{1}{2}$ times the circumference times the radius.

In the previous activity, students had experience following along as the teacher developed the justification. This time give students the opportunity to write their own justification for the area of a circle. As students work, monitor and select students who have clear, but different, explanations to share during the whole-group discussion. In particular, select students who use the following steps:

- $\text{Area} = \frac{1}{2} \boldsymbol{\cdot} \text{base} \boldsymbol{\cdot} \text{height}$
- $\text{Area} = \frac{1}{2} \boldsymbol{\cdot} \text{circumference} \boldsymbol{\cdot} \text{radius}$
- $\text{Area} = \frac{1}{2} \boldsymbol{\cdot} (\pi d) \boldsymbol{\cdot} r$
- $\text{Area} = \pi r \boldsymbol{\cdot} r$
- $\text{Area} = \pi r^2$

If the bands making up the circle really did not stretch, then they would not form rectangles when they are unwound because the circumference of the inner circle is not the same as the circumference of the outer circle in each band. A rectangle is an appropriate approximation for the shape in terms of calculating its area.

Addressing

- 7.G.B.4

Instructional Routines

- MLR1: Stronger and Clearer Each Time

- Think Pair Share

Launch

Give students quiet work time followed by partner and whole-class discussion.

Access for Students with Disabilities

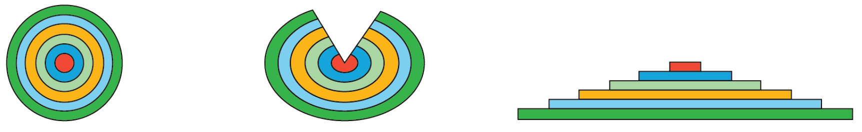
Representation: Internalize Comprehension. Activate or supply background knowledge about finding the area of triangles and circles. Allow students to use calculators to ensure inclusive participation in the activity.
Supports accessibility for: Memory; Conceptual processing

Anticipated Misconceptions

If students struggle to imagine the circle and how it is cut and rearranged, suggest a familiar material for the rings that bends but does not stretch (for example, a cord or chain).

Student Task Statement

Imagine a circle made of rings that can bend, but not stretch.



A circle is made of rings. The rings are unrolled. The circle has been made into a new shape.

1. What polygon does the new shape resemble?
2. How does the area of the polygon compare to the area of the circle?
3. How can you find the area of the polygon?
4. Show, in detailed steps, how you could find the polygon's area in terms of the circle's measurements. Show your thinking. Organize it so it can be followed by others.
5. After you finish, trade papers with a partner and check each other's work. If you disagree, work to reach an agreement. Discuss:
    - Do you agree or disagree with each step?
    - Is there a way to make the explanation clearer?
6. Return your partner's work, and revise your explanation based on the feedback you received.

Student Response

1. A triangle

2. The areas of the shapes are equal.
3. It looks like a triangle, so the area can be found with the formula $\text{Area} = \frac{1}{2} \boldsymbol{\cdot} \text{base} \boldsymbol{\cdot} \text{height}$.
4. The base of the "triangle" has length equal to the circumference of the circle, while its height is the radius of the circle. So:
    - $\text{Area} = \frac{1}{2} \boldsymbol{\cdot} \text{circumference} \boldsymbol{\cdot} \text{radius}$
    - $\text{Area} = \frac{1}{2} \boldsymbol{\cdot} \pi d \boldsymbol{\cdot} r$
    - $\text{Area} = \pi r \boldsymbol{\cdot} r$
    - $\text{Area} = \pi r^2$

Activity Synthesis

Ask selected students to explain their steps for finding the area in terms of the circle's measures. Ask the class whether they agree, disagree, or have questions after each student shares their reasoning.

Access for English Language Learners

Writing, Speaking, Listening: MLR1 Stronger and Clearer Each Time. After students have the opportunity to think about how to find the area of the triangle in terms of the circle's measurements, ask students to write a brief explanation of their process. As students prepare their explanation, look for students who state that the area of the shape that resembles a triangle is half of the circumference multiplied by the radius of the circle. Ask each student to meet with 2–3 other partners for feedback. Provide students with prompts for feedback that will help them strengthen their ideas and clarify their language (e.g., "Can you explain how…," "You should expand on…," etc.). Students can borrow ideas and language from each partner to refine and clarify their original explanation. This will help students revise and refine both their ideas and their verbal and written output.
Design Principles(s): Optimize output (for explanation); Maximize meta-awareness

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8.4 Tiling a Table

5 minutes
The purpose of this activity is for students to apply the formula for area of a circle to solve a problem in context. The diameter of the circle is given, so students must first determine the radius.

Addressing

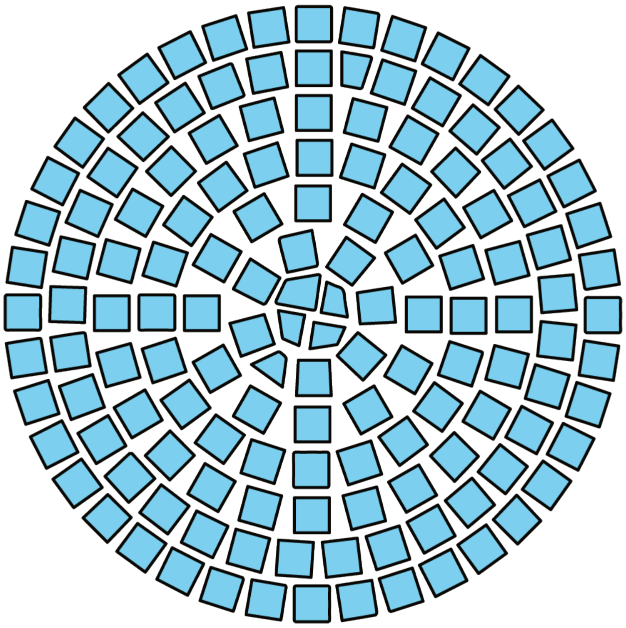
- 7.G.B.4

Instructional Routines

- Notice and Wonder

Launch

Display this image of table top for all to see. Ask students, “What do you notice? What do you wonder?”



Quiet work time followed by a whole-class discussion.

Anticipated Misconceptions

Students may square the diameter, forgetting that they need to determine the radius first.

> ### Student Task Statement
>
> Elena wants to tile the top of a circular table. The diameter of the table top is 28 inches. What is its area?

Student Response

The area of the circular table is about 615 in^2, because the diameter 28 in gives a radius of 14 in, and $\pi \cdot 14^2 \approx 615$.

> ### Are You Ready for More?
>
> A box contains 20 square tiles that are 2 inches on each side. How many boxes of tiles will Elena need to tile the table?

Student Response

$615 \div 4 = 153.75$. She would definitely have enough tiles from 8 boxes, and 7 boxes would probably be enough because of the space left in between the tiles for grout. It might be necessary to cut some of the tiles, especially near the boundary, so that they don't hang over the edge of the table.

Activity Synthesis

Invite students to share their strategies for finding the area of the tabletop. After 3 students have shared their strategies, ask the class what formula all of the students used for finding the area of a circle. Record and display this formula, $A = \pi r^2$, for all to see. Here, A is the area of the circle, and r is the radius of the circle.

Access for Students with Disabilities

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation such as: “First, I _____ because....”, “I noticed _____ so I....”, “Why did you...?”, and “I agree/disagree because....”
Supports accessibility for: Language; Social-emotional skills

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Lesson Synthesis

The main ideas are:

- We can find the area of a circle if we know the radius or the diameter.
- We know that the radius is half the length of the diameter.
- The formula for finding area of a circle is $A = \pi r^2$.

Discussion Questions:

- How would you find the area of a circle with a radius of 10? (Multiply π times 100, because $10^2 = 100$.)
- How would you find the area of a circle with a diameter of 10? (Multiply π times 25, because $10 \div 2 = 5$ and $5^2 = 25$.)

8.5 A Circumference of 44

Cool Down: 5 minutes

Addressing

- 7.G.B.4

Anticipated Misconceptions

Since the answers to questions 2 and 3 are dependent on the answer to question 1, check that students have accurately determined the diameter, and if necessary, remind them that since circumference is a little more than 3 times as long as the diameter, then the diameter is a little less than $\frac{1}{3}$ of the circumference.

Student Task Statement

A circle's circumference is approximately 44 cm. Complete each statement using one of these values:

7, 11, 14, 22, 88, 138, 154, 196, 380, 616.

1. The circle's diameter is approximately ________ cm.
2. The circle's radius is approximately ________ cm.
3. The circle's area is approximately ________ cm^2.

Student Response

1. 14
2. 7
3. 154

Student Lesson Summary

If C is a circle's circumference and r is its radius, then $C = 2\pi r$. The area of a circle can be found by taking the product of half the circumference and the radius.

If A is the area of the circle, this gives the equation: $A = \frac{1}{2}(2\pi r) \boldsymbol{\cdot} r$

This equation can be rewritten as: $A = \pi r^2$

(Remember that when we have $r \boldsymbol{\cdot} r$ we can write r^2 and we can say "r **squared**.")

This means that if we know the radius, we can find the area. For example, if a circle has radius 10 cm, then the area is about $(3.14) \boldsymbol{\cdot} 100$ which is 314 cm^2.

If we know the diameter, we can figure out the radius, and then we can find the area. For example, if a circle has a diameter of 30 ft, then the radius is 15 ft, and the area is about $(3.14) \boldsymbol{\cdot} 225$ which is approximately 707 ft^2.

Glossary

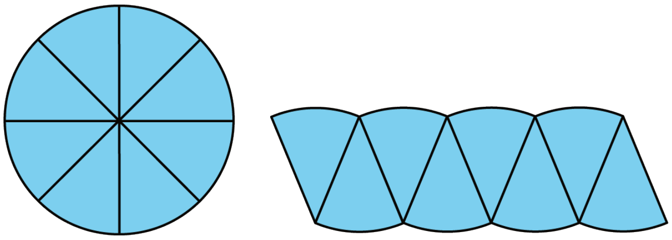
- squared

Lesson 8 Practice Problems

Problem 1

Statement

The picture shows a circle divided into 8 equal wedges which are rearranged.



The radius of the circle is r and its circumference is $2\pi r$. How does the picture help to explain why the area of the circle is πr^2?

Solution

The rearranged shape looks more and more like a rectangle as the circle is cut into more pieces. The length of the rectangle is about half of the circumference of the circle or πr, and its height is roughly the radius r. So the area of the rectangle (and of the circle) is πr^2.

Problem 2

Statement

A circle's circumference is approximately 76 cm. Estimate the radius, diameter, and area of the circle.

Solution

The radius is approximately 12 cm. The diameter is approximately 24 cm. The area is approximately 460 cm^2.

Problem 3

Statement

Jada paints a circular table that has a diameter of 37 inches. What is the area of the table?

Solution

About 1,075 in^2

Problem 4

Statement

The Carousel on the National Mall has 4 rings of horses. Kiran is riding on the inner ring, which has a radius of 9 feet. Mai is riding on the outer ring, which is 8 feet farther out from the center than the inner ring is.

a. In one rotation of the carousel, how much farther does Mai travel than Kiran?

b. One rotation of the carousel takes 12 seconds. How much faster does Mai travel than Kiran?

Solution

a. about $106.8 - 56.5$, or 50.3 feet farther

b. about $50.3 \div 12$, or 4.2 feet per second faster

(From Unit 3, Lesson 4.)

Problem 5

Statement

Here are the diameters of four coins:

coin	penny	nickel	dime	quarter
diameter	1.9 cm	2.1 cm	1.8 cm	2.4 cm

a. A coin rolls a distance of 33 cm in 5 rotations. Which coin is it?

b. A quarter makes 8 rotations. How far did it roll?

c. A dime rolls 41.8 cm. How many rotations did it make?

Solution

a. Nickel because $33 \div 5 \div \pi \approx 2.1$

b. About 60.3 cm because $2.4 \boldsymbol{\cdot} \pi \boldsymbol{\cdot} 8 \approx 60.3$

c. About 7 because $41.8 \div \pi \div 1.8 \approx 7$

(From Unit 3, Lesson 5.)