Lesson 9: Applying Area of Circles

Goals

- Calculate the area of a shape that includes circular or semi-circular parts, and explain (orally and in writing) the solution method.
- Comprehend and generate expressions in terms of π to express exact measurements related to a circle.

Learning Targets

- I can calculate the area of more complicated shapes that include fractions of circles.
- I can write exact answers in terms of π.

Lesson Narrative

In previous lessons, students estimated the area of circles on a grid and explored the relationship between the circumference and the area of a circle to see that $A = \pi r^2$. In this lesson, students apply this formula to solve problems involving the area of circles as well as shapes made up of parts of circles (MP 1 and 2) and other shapes such as rectangles. These calculations require composition and decomposition recalling work from grade 6.

Also, in this lesson students are introduced to the idea of expressing exact answers in terms of π.

Alignments

Addressing

- 7.G.B.4: Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR1: Stronger and Clearer Each Time
- MLR7: Compare and Connect
- Think Pair Share

Required Materials

Four-function calculators

Required Preparation

It is recommended that four-function calculators be made available to take the focus off computation.

Student Learning Goals

Let’s find the areas of shapes made up of circles.

9.1 Still Irrigating the Field

Warm Up: 5 minutes

The purpose of this activity is for students to calculate a more exact answer to a problem from the previous lesson in which they estimated the area. Each answer choice listed results from using a different approximation of π.

Addressing

- 7.G.B.4

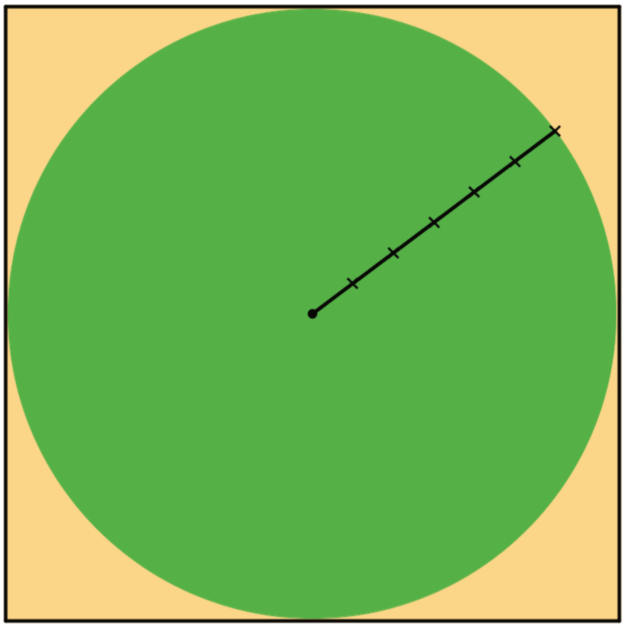
Launch

Remind students that the circular field is enclosed by a square that is 800 m on a side. If students ask what approximation they should use for π, tell them they can choose.

Give students 2 minutes of quiet work time followed by a whole-class discussion.

Student Task Statement

The area of this field is about 500,000 m^2. What is the field’s area to the nearest square meter? Assume that the side lengths of the square are exactly 800 m.



- 502,400 m^2
- 502,640 m^2
- 502,655 m^2

- 502,656 m^2
- 502,857 m^2

Student Response

Answers vary based on the chosen approximation for π. The most accurate answer is C, 502,655 m^2.

Activity Synthesis

All the answer choices are possible, but because the radius of the circle is so large, using a more approximate value for π can lead to a noticeable rounding error.

- $3.14 \cdot 400^2 = 502{,}400$
- $3.1415 \cdot 400^2 = 502{,}640$
- $3.1415927 \cdot 400^2 \approx 502{,}655$
- $3.1416 \cdot 400^2 = 502{,}656$
- $\frac{22}{7} \cdot 400^2 \approx 502{,}857$

The most accurate answer, $502{,}655 \text{ m}^2$, comes from using at least 6 decimal places for π.

9.2 Comparing Areas Made of Circles

20 minutes

The purpose of this activity is for students to find the areas of regions involving different-sized circles and compare the strategies used. The first question introduces subtraction as a strategy to find the area around the outside of a circle. The second question introduces division to find the area of fractions of a circle.

Monitor for students who use different strategies for finding the area, including:

- Calculating 30.96 square units for Figure A, $4 \cdot 7.74$ for Figure B, and $9 \cdot 3.44$ for Figure C
- Realizing that all 3 figures end up being $144 - 113.04$ so their areas had to be equal
- Noticing that Figure B is composed of 4 scaled copies of Figure A, each with a scale factor of $\frac{1}{2}$ and therefore $\frac{1}{4}$ as much area. Figure C is composed of 9 scaled copies of Figure A, each with a scale factor of $\frac{1}{3}$ and therefore $\frac{1}{9}$ as much area. (If a student does not make this realization, it is *not* necessary for the teacher to bring it up during the discussion.)
- Calculating 8.28 square units for Figure D and 8.71 square units for Figure E
- Realizing that Figures D has 2 fewer squares and 2 more quarter-circles (which are smaller than the squares) so it must have a smaller area

The task affords an opportunity for students to engage in MP6. Some students may leave the area expressions in terms of π, but others may use an approximation for π. It is very important, particularly for the first problem, that they use the *same* approximation for π in all 3 figures. Otherwise they will end up with numerically different answers.

Addressing

- 7.G.B.4

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR1: Stronger and Clearer Each Time
- Think Pair Share

Launch

Arrange students in groups of 2. Display the image in the first question and ask students to make a prediction before calculating. Give 30 seconds of quiet think time before sharing with their partner. Give students quiet work time followed by a whole-class discussion.

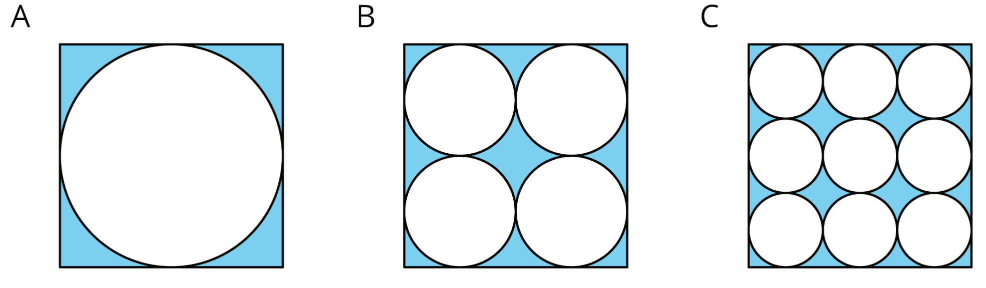
Anticipated Misconceptions

In the first question, students may not know how to find the radius of the circles. Suggest having them cut off the shaded regions and rearrange them to show that the length of each side fits half way across the circle (marking the radius).

In the second question, students might benefit from cutting and rearranging the figures. Some students might assume, based on previous activities, that the areas of both figures are equal. However, Figure D has more pieces that are parts of a circle, and Figure E has more units that are a full square. Ask students whether the fourth of the circle has the same area as the square.

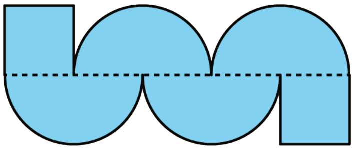
Student Task Statement

1. Each square has a side length of 12 units. Compare the areas of the shaded regions in the 3 figures. Which figure has the largest shaded region? Explain or show your reasoning.

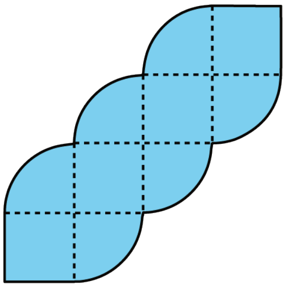


2. Each square in Figures D and E has a side length of 1 unit. Compare the area of the two figures. Which figure has more area? How much more? Explain or show your reasoning.

D



E



Student Response

1. The areas of all 3 shaded regions are equal: about 30.96 square units (144 - 36π square units). The area of the entire square is 144 square units, because $12 \cdot 12 = 144$. The area of the 1 large circle is approximately 113.04 square units, because $3.14 \cdot 6^2 = 113.04$. The area of the 4 medium circles is also 113.04 square units, because $3.14 \cdot 3^2 = 28.26$, and $28.26 \cdot 4 = 113.04$. The area of the 9 small circles is also 113.04 square units, because $3.14 \cdot 2^2 = 12.56$, and $12.56 \cdot 9 = 113.04$. The area of the shaded region in each figure is 30.96 square units, because $144 - 113.04 = 30.96$.

2. Figure E's area is about 0.43 square units larger than Figure D's. Figure D consists of 2 squares and 4 half circles, giving it an area of about 8.28 square units, because $4 \cdot 1.57 + 2 = 8.28$. Figure E consists of 4 squares and 6 quarter circles, giving it an area of about 8.71 square units, because $6 \cdot 0.785 + 4 = 8.71$.

Are You Ready for More?

Which figure has a longer perimeter, Figure D or Figure E? How much longer?

Student Response

Figure D's perimeter is $\pi + 2$ units longer than Figure E's because $(4\pi + 6) - (3\pi + 4) = 1\pi + 2$.

Activity Synthesis

There are two main goals for this discussion: for students to notice ways to be more efficient when comparing the areas of the regions and to be introduced to expressing answers in terms of pi.

Display Figures A, B, and C for all to see. Ask selected students to share their reasoning. Sequence the strategies from most calculations to most efficient.

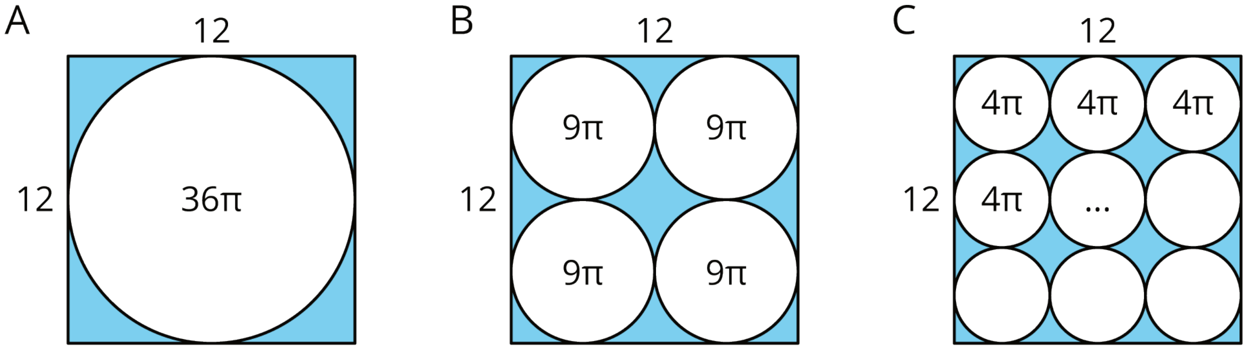
If there were selected students who determined the areas were equal before calculating, ask them to share how they could tell. If there were no selected students, ask the class how we could determine that the area of the shaded regions in Figures A, B, and C were equal before calculating the answer of 30.96.

Next, focus the discussion on leaving answers in terms of π for each figure. Explain to students that in Figure A, the radius of the circle is 6, so the area of the circular region is $\pi \cdot 6^2$. Instead of multiplying by an approximation of π, we can express this answer as 36π. This is called answering *in terms of* π. Consider writing "36π" inside the large circle of Figure A.

Discuss:

- In terms of π, what is the area of one of the circular regions in Figure B? (9π)
- What is the combined area of all four circles in Figure B? ($4 \cdot 9\pi$, or 36π)
- What is the area of one of the circular regions in Figure C? (4π)
- What is the combined area of all nine circles in Figure C? ($9 \cdot 4\pi$, or 36π)

Consider writing "9π" and "4π" inside some of the circles in Figures B and C. Explain that the area of the shaded region for each of these figures is $144 - 36\pi$.



Discuss how students' strategies differed between the first problem (about Figures A, B, and C) and the second problem (about Figures D and E) and why.

Ask students to express the area of Figures D and E in terms of π. Record and display their answers of $2 + 2\pi$ and $4 + 1.5\pi$ for all to see. Ask students to discuss how they can tell Figure E's area is larger than Figure D's area when they are both written in terms of π.

Access for English Language Learners

Writing, Speaking, Listening: MLR1 Stronger and Clearer Each Time. After students have determined which figure has the largest shaded region, ask students to show their work and provide a brief explanation of their reasoning. Ask each student to meet with 2–3 other partners for feedback. Provide students with prompts for feedback that will help them strengthen their ideas and clarify their language (e.g., "How did you find the radius of each circle in the figure?", "Why did you . . . ?", etc.). Students can borrow ideas and language from each partner to refine and clarify their original explanation. This will help students refine their own explanation and learn about different strategies to find area.
Design Principles(s): Optimize output (for explanation); Maximize meta-awareness

9.3 The Running Track Revisited

Optional: 10 minutes

Earlier in this lesson, students found the area of regions around circles and the area of fractions of circles in separate problems. In this activity, students combine these two strategies to find the area of a complex real-world object. Students engage in MP2 as they decide how to decompose the running track into measurable pieces and how to use the given information about the dimensions of the track to calculate areas.

Addressing

- 7.G.B.4

Instructional Routines

- MLR7: Compare and Connect

Launch

Arrange students in groups of 2. Give students 3–4 minutes of partner work time followed by small-group and whole-class discussions.

Access for Students with Disabilities

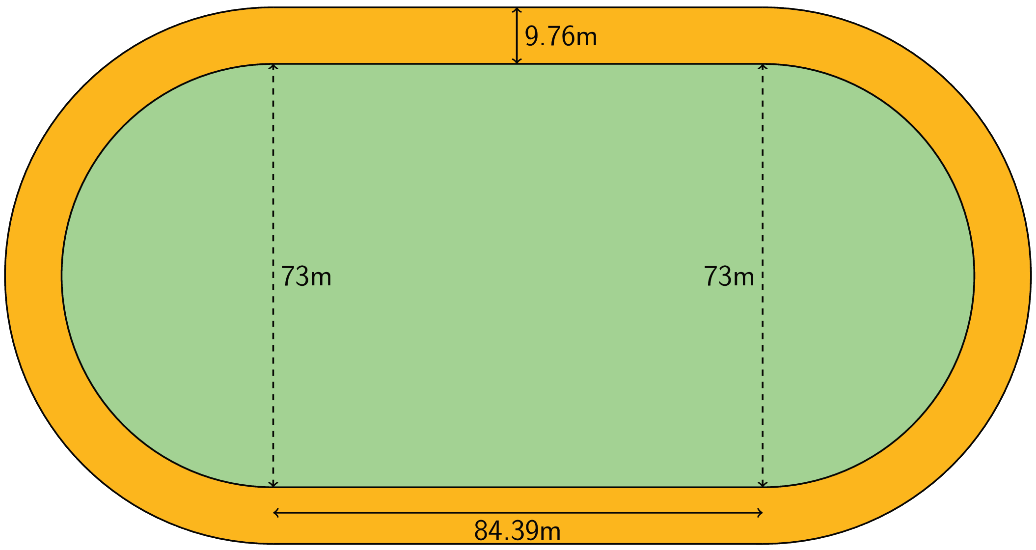
Action and Expression: Internalize Executive Functions. Provide students with a graphic organizer to organize the information provided in the problem and to structure their problem-solving strategy. The graphic organizer should include the prompts: “What do I need to find out?”, “What do I need to do?”, “How I solved the problem.”, and “How I know my answer is correct.”
Supports accessibility for: Language; Organization

Anticipated Misconceptions

Some students may think they can calculate the area of the running track by multiplying half of the perimeter times the radius, as if the shape were just a circle. Prompt them to see that they need to break the overall shape into rectangular and circular pieces.

Student Task Statement

The field inside a running track is made up of a rectangle 84.39 m long and 73 m wide, together with a half-circle at each end. The running lanes are 9.76 m wide all the way around.



What is the area of the running track that goes around the field? Explain or show your reasoning.

Student Response

The area of the entire running track is $807.7376\pi + 823.6464$, or about $4{,}183.6 \text{ m}^2$. The area of the straight top and bottom of the running track are each 823.6464 m^2 because $84.39 \cdot 9.76 = 823.6464$. The left and right sides of the running track are half circles with a smaller half circle missing from the inside. The area of the inside circle is about $4{,}183.265 \text{ m}^2$ because $36.5^2 \cdot 3.14 = 4{,}183.265$. The area of the outside circle is about $6{,}719.561064 \text{ m}^2$ because $36.5 + 9.76 = 46.26$ and $46.26^2 \cdot 3.14 = 6{,}719.561064$. The area of each curved side of the running track is about $1{,}268.148032 \text{ m}^2$ because $(6{,}719.561064 - 4{,}183.265) \div 2 = 1{,}268.148032$. The area of the entire running track is about $4{,}183.6 \text{ m}^2$ because $2 \cdot 823.6464 + 2 \cdot 1{,}268.148032 = 4{,}183.588864$.

Activity Synthesis

Pair the groups of 2 to create groups of 4. Have students compare answers and explain their reasoning until they reach an agreement.

As a whole group, discuss how finding the area of the track in this activity was similar or different to solving the two area problems in the previous activity. Make sure to highlight these points:

- Like the first problem in the previous activity, finding the area of the track can be done by finding the area of the larger shape (the track and the field inside) and then taking away the area of the field inside.
- Like the second problem in the previous activity, there are semicircles whose area can be found by composing them to make a full circle or by taking half the area of the corresponding full circle.

Access for English Language Learners

Representing, Speaking, Listening: MLR7 Compare and Connect. Ask students to prepare a visual display that shows how they calculated the area of the running track that goes around the field, and look for students with different strategies for calculating the area of the curved parts of the track. As students investigate each other's work, ask students to share what worked well in a particular approach. During this discussion, listen for and amplify any comments that clarify that the curved parts of the running track are half circles with a smaller circle missing from the inside. Then encourage students to make connections between the quantities representing areas and the shapes in the diagram. Ask questions such as, "What does the quantity 6,719.561 represent in the diagram?" and "What does the quantity 4,183.265 represent in the diagram?" During this discussion, listen for and amplify language students use to reason that these quantities represent the areas of larger and smaller circles, respectively. This will foster students' meta-awareness and support constructive conversations as they compare strategies for finding the area of a complex real-world object.
Design Principle(s): Cultivate conversation; Maximize meta-awareness

Lesson Synthesis

Discussion Questions:

- What is the area, in terms of π, of a circle with a radius of 10? (100π, because $10^2 = 100$.)
- What is the area, in terms of π, of a circle with a diameter of 10? (25π, because $10 \div 2 = 5$ and $5^2 = 25$.)
- What is the area, in terms of π, of a half-circle with a diameter of 10? (12.5π, because $25 \div 2 = 12.5$.)

9.4 Area of an Arch

Cool Down: 5 minutes
This cool-down purposefully does not specify whether students should give an exact answer (in terms of pi) or use an approximation. This ambiguity provides an opportunity for teachers to assess whether students have internalized that leaving their answer in terms of pi is an acceptable way to express an answer when they aren't told what to round to or what approximation to use.

Addressing

- 7.G.B.4

Anticipated Misconceptions

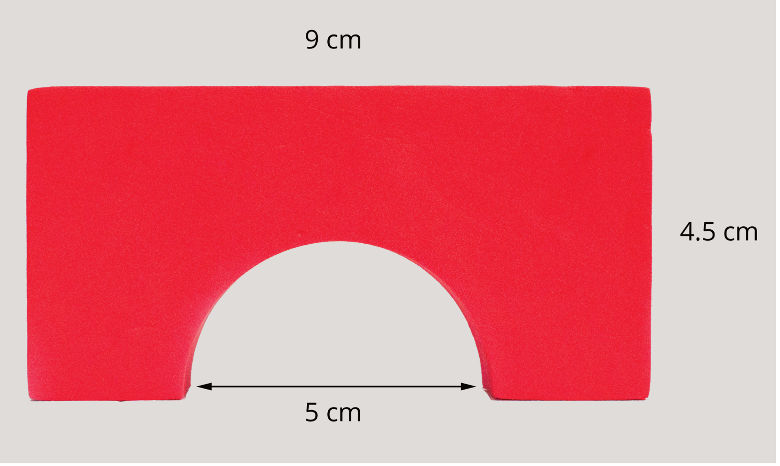
Students may think the word side refers to the length of the outer sides in the block. Tell these students that side, in this context, refers to the face of the block they are given.

Students might correctly find the areas of the rectangle and the half-circle but add these values instead of subtracting.

Students might forget to divide the area of the circle by 2 to find the area of the half-circle.

Student Task Statement

Here is a picture that shows one side of a child's wooden block with a semicircle cut out at the bottom.



Find the area of the side. Explain or show your reasoning.

Student Response

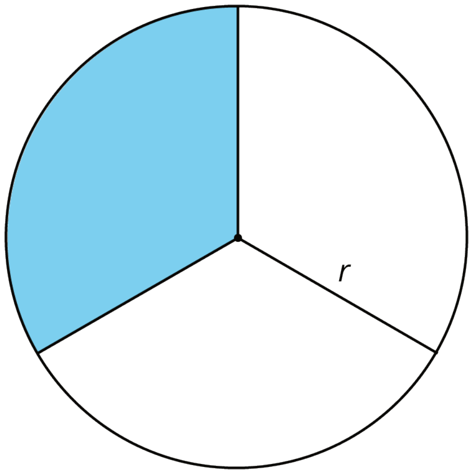
The area of the side of the block is about 30.68 cm^2. The area of the rectangle is $9 \cdot 4.5$, or 40.5 cm^2. The area of a circle with a diameter of 5 cm is $6.25\pi \text{ cm}^2$. The front face of the wooden block is a rectangle missing half of circle with diameter 5 cm, so its area in cm^2 is $40.5 - 3.125\pi$ or about 30.68.

Student Lesson Summary

The relationship between A, the area of a circle, and r, its radius, is $A = \pi r^2$. We can use this to find the area of a circle if we know the radius. For example, if a circle has a radius of 10 cm, then the area is $\pi \cdot 10^2$ or 100π cm^2. We can also use the formula to find the radius of a circle if we know the area. For example, if a circle has an area of 49π m^2 then its radius is 7 m and its diameter is 14 m.

Sometimes instead of leaving π in expressions for the area, a numerical approximation can be helpful. For the examples above, a circle of radius 10 cm has area about 314 cm^2. In a similar way, a circle with area 154 m^2 has radius about 7 m.

We can also figure out the area of a fraction of a circle. For example, the figure shows a circle divided into 3 pieces of equal area. The shaded part has an area of $\frac{1}{3}\pi r^2$.



Lesson 9 Practice Problems

Problem 1

Statement

A circle with a 12-inch diameter is folded in half and then folded in half again. What is the area of the resulting shape?

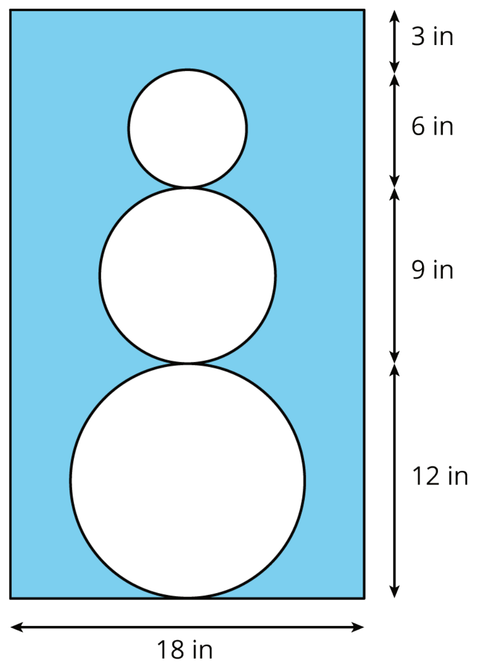
Solution

9π in^2, or about 28 in^2, because $\frac{1}{4} \cdot 6^2\pi = 9\pi$

Problem 2

Statement

Find the area of the shaded region. Express your answer in terms of π.



Solution

$540 - 65.25\pi \text{ in}^2$. Find the area of the rectangle by multiplying $18 \cdot 30 = 540$. Find the radii of the circles, square them, and add them together. $6^2 + 4.5^2 + 3^2 = 65.25$. Multiply 65.25 by π to get the total area of the circles. Subtract 65.25π from 540 to find the area of the shaded region.

Problem 3

Statement

The face of a clock has a circumference of 63 in. What is the area of the face of the clock?

Solution

About 316 in^2. Divide 63 by π and by 2 to determine the radius of the clock. $63 \div 2 \div \pi \approx 10$. To find the area of the face of the clock multiply π by 10^2.

(From Unit 3, Lesson 8.)

Problem 4

Statement

Which of these pairs of quantities are proportional to each other? For the quantities that are proportional, what is the constant of proportionality?

a. Radius and diameter of a circle

b. Radius and circumference of a circle

c. Radius and area of a circle

d. Diameter and circumference of a circle

e. Diameter and area of a circle

Solution

a. Yes. The diameter is twice the radius so the constant of proportionality is either 2 or $\frac{1}{2}$.

b. Yes. The circumference is 2π times the radius so the constant of proportionality is either 2π or $\frac{1}{2\pi}$.

c. No

d. Yes. The circumference is π times the diameter so the constant of proportionality is either π or $\frac{1}{\pi}$.

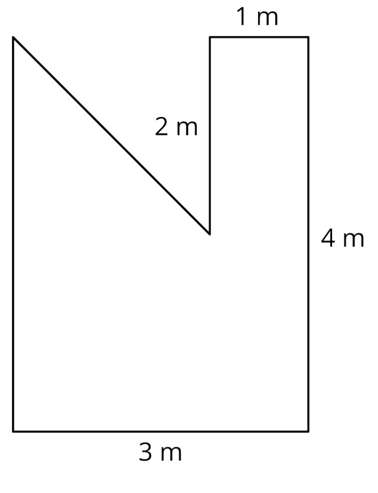
e. No

(From Unit 3, Lesson 7.)

Problem 5

Statement

Find the area of this shape in two different ways.



Solution

10 m^2. Explanations vary. Sample responses:

a. It is a rectangle of area 12 m^2 with a triangle of area 2 m^2 missing.

b. It is a rectangle of area 6 m^2 plus a rectangle of area 2 m^2 plus a triangle of area 2 m^2.

(From Unit 3, Lesson 6.)

Problem 6

Statement

Elena and Jada both read at a constant rate, but Elena reads more slowly. For every 4 pages that Elena can read, Jada can read 5.

a. Complete the table.

pages read by Elena	pages read by Jada
4	5
1	
9	
e	
	15
	j

b. Here is an equation for the table: $j = 1.25e$. What does the 1.25 mean?

c. Write an equation for this relationship that starts $e = ...$

Solution

a.

pages read by Elena	pages read by Jada
4	5
1	$\frac{5}{4}$ or 1.25
9	$\frac{45}{4}$ or 11.25
e	$\frac{5}{4}e$ or $1.25e$
12	15
$\frac{4}{5}j$ or $0.8j$	j

b. For every one page that Elena reads, Jada reads 1.25 pages.

c. $e = \frac{4}{5}j$ or $e = 0.8j$

(From Unit 2, Lesson 5.)