Lesson 14: Decimal Representations of Rational Numbers

Goals

- Comprehend that a rational number is a fraction or its opposite, and that a rational number can be represented with a decimal expansion that “repeats” or “terminates”.
- Represent rational numbers as equivalent decimals and fractions, and explain (orally) the solution method.

Learning Targets

- I can write a fraction as a repeating decimal.
- I understand that every number has a decimal expansion.

Lesson Narrative

In the last two lessons in this unit, students explore decimal representations of rational and irrational numbers. The zooming number line representation used in these lessons supports students' understanding of place value and helps them form mental images of the two different ways a decimal expansion may go on forever (depending on whether the number is rational or irrational).

This first lesson explores the different forms of rational numbers. The warm-up reviews the idea of rational numbers as fractions of the form $\frac{a}{b}$ using tape diagrams. The first classroom activity, which is optional, continues with the same fractions by writing them as decimals.

In the second classroom activity students work with a variety of rational numbers written in different forms, including fractions, decimals and square roots. They see that it is not the symbols used to write a number that makes it rational but rather the fact that it can be rewritten in the form $\frac{a}{b}$ where a and b are integers, e.g. $\sqrt{\frac{1}{9}} = \frac{1}{3}$.

In the last activity students explore the decimal expansion of $\frac{2}{11}$. They use long division with repeated reasoning (MP8) to find that $\frac{2}{11} = 0.1818\ldots$. Students realize that they could easily keep zooming in on $\frac{2}{11}$ because of the pattern of alternating between the intervals $\frac{1}{10}$–$\frac{2}{10}$ and $\frac{8}{10}$–$\frac{9}{10}$ of the previous line. The goal is for students to notice and appreciate the predictability of repeating decimals and see how that connects with the $\frac{a}{b}$ structure.

By the end of this lesson students have seen that rational numbers can have decimal representations that terminate or that eventually repeat. This begs the question if there are numbers with non-terminating decimal representations that do not repeat. This leads into the next lesson.

Alignments

Building On

- 7.NS.A.2.d: Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Addressing

- 8.EE.A: Work with radicals and integer exponents.
- 8.NS.A: Know that there are numbers that are not rational, and approximate them by rational numbers.
- 8.NS.A.1: Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

Building Towards

- 8.NS.A: Know that there are numbers that are not rational, and approximate them by rational numbers.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR1: Stronger and Clearer Each Time
- MLR3: Clarify, Critique, Correct
- Notice and Wonder
- Think Pair Share

Student Learning Goals

Let's learn more about how rational numbers can be represented.

14.1 Notice and Wonder: Shaded Bars

Warm Up: 5 minutes

The purpose of this warm-up is for students to review what rational numbers are—fractions and their opposites. In earlier lessons, students explored square and cube roots and now return to strictly rational numbers. While students do not need to be able to recite a definition for rational numbers, this type of diagram is one students are likely familiar with and gives students an opportunity to use related language such as part, whole, halves, 1 out of 8 parts, one eighth, etc. After students work with different representations of rational numbers in upcoming lessons, they will expand their definition and understanding of irrational numbers.

Building Towards

- 8.NS.A

Instructional Routines

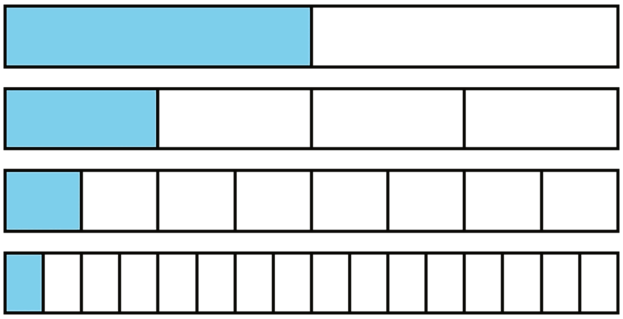
- Notice and Wonder

Launch

Arrange students in groups of 2. Tell students that they will look at an image, and their job is to think of at least one thing they notice and at least one thing they wonder. Display the image for all to see. Ask students to give a signal when they have noticed or wondered about something. Give students 1 minute of quiet think time, and then 1 minute to discuss the things they notice with their partner, followed by a whole-class discussion.

Student Task Statement

What do you notice? What do you wonder?



Student Response

Answers vary. Possible responses:

Students may notice:

- The highlighted areas can be written as fractions, parts of wholes.
- Each bar is broken into equal-sized parts.
- They look like tape diagrams broken into halves, fourths, eighths, sixteenths, etc.
- The whole could be any value, not necessarily 1.
- If we knew the whole, we could find the value of any of the highlighted areas by finding half of it, a fourth of it, etc.

Students may wonder:

- What is the value of the whole?
- What is the value of each highlighted area?

Activity Synthesis

Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If possible, record the relevant reasoning on or near the image. After each response, ask the class if they agree or disagree and to explain alternative ways of thinking, referring back to the images each time.

14.2 Halving the Length

Optional: 10 minutes

The purpose of this optional activity is for students to see what the figure in the warm up didn't explore: as 1 is successively divided in half, the number of digits needed to accurately describe the resulting value increases. Students likely have memorized that a half is 0.5 and a fourth is 0.25, but what is half of 0.25? How can you calculate it? Use this activity if you think your students need a reminder about place value or long division with decimals.

Addressing

- 8.NS.A

Instructional Routines

- MLR1: Stronger and Clearer Each Time

Launch

Do not provide access to calculators. Give students 2 minutes quiet work time followed by partner then whole-class discussion.

Access for English Language Learners

Writing, Speaking: MLR1 Stronger and Clearer Each Time. Use this routine to give students a structured opportunity to revise and refine their response to: "How do the decimal representations change?" Ask each student to meet with 2–3 other partners for feedback. Display prompts for feedback that will help students strengthen their ideas and clarify their language. For example, "Can you explain how. . . ?", "You should expand on. . . .", etc. Give students 1–2 minutes to revise their writing based on the feedback they received.
Design Principle(s): Optimize output (for generalization)

Student Task Statement

Here is a number line from 0 to 1.

1. Mark the midpoint between 0 and 1. What is the decimal representation of that number?
2. Mark the midpoint between 0 and the newest point. What is the decimal representation of that number?
3. Repeat step two. How did you find the value of this number?
4. Describe how the value of the midpoints you have added to the number line keep changing as you find more. How do the decimal representations change?

Student Response

1. The decimal representation of the midpoint between 0 and 1 is 0.5.
2. The decimal representation of the midpoint between 0 and 0.5 is 0.25.
3. The decimal representation of the midpoint between 0 and 0.25 is 0.125. Answers vary. Sample response: I divided 0.25 by 2 using long division.
4. Answers vary. Sample response: the value of the midpoint keeps halving while the number of digits needed to write the value of the midpoint increases by 1 each time.

Activity Synthesis

Discuss:

- "How can you use long division to answer the third problem?"
- "What are these values when written in fraction notation? How do you know?"

Access for Students with Disabilities

Representation: Develop Language and Symbols. Create a display of important terms and vocabulary. Include the following terms and maintain the display for reference throughout the unit: decimal form and fraction form. Provide step by step instructions for how to convert fractions to decimals and vice versa. Be sure to provide an example using long division.
Supports accessibility for: Memory; Language

14.3 Recalculating Rational Numbers

20 minutes
The purpose of this task is for students to rewrite rational numbers with terminating decimal expansions in fraction form and fractions with terminating decimal expansions as decimals. This activity is the first of a series of three in which students rewrite numbers in different ways, supporting their understanding of what rational and irrational numbers are and how they can be represented.

Monitor for students who write $\frac{1}{5}$ and $\frac{2}{10}$ for 0.2.

Addressing

- 8.EE.A
- 8.NS.A

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR3: Clarify, Critique, Correct

Launch

Arrange students in groups of 2. Do not provide access to calculators.

Remind students that all rational numbers are fractions and their opposites, and they can all be written in the form $\frac{a}{b}$ where a and b are integers (with $b \neq 0$). In fact, there are many equivalent fractions that represent a single rational number. For example, 5 is equivalent to $\frac{10}{2}$ and $\frac{15}{3}$.

Students should complete the problems individually, then compare with their partners and come to resolutions over any differences. Repeat this process for the second problem.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. To support development of organizational skills, check in with students within the first 2-3 minutes of work time. Look for students who find equivalent ways to represent fractions and decimals. Check for precision in calculations and encourage partners to seek clarification in their discussions.
Supports accessibility for: Memory; Organization

Access for English Language Learners

Writing: MLR3 Clarify, Critique, Correct. Present an incorrect statement that reflects a possible misunderstanding from the class. For example, finding the decimal representations of rational numbers, an incorrect statement is: “For finding the decimal representation of $\frac{3}{8}$, divide 8 by 3 using long division ”. Prompt students to identify the error, and then write a correct version. This helps students evaluate, and improve on, the written mathematical arguments of others.
Design Principle(s): Maximize meta-awareness

Anticipated Misconceptions

Some students may write that $0.333 = \frac{1}{3}$. If so, ask them to check their work by calculating the decimal representation of $\frac{1}{3}$.

Student Task Statement

1. Rational numbers are fractions and their opposites. All of these numbers are rational numbers. Show that they are rational by writing them in the form $\frac{a}{b}$ or -$\frac{a}{b}$.

   a. 0.2

   b. -$\sqrt{4}$

   c. 0.333

   d. $\sqrt[3]{1000}$

   e. -1.000001

   f. $\sqrt{\frac{1}{9}}$

2. All rational numbers have decimal representations, too. Find the decimal representation of each of these rational numbers.

   a. $\frac{3}{8}$

   b. $\frac{7}{5}$

   c. $\frac{999}{1000}$

   d. $\frac{111}{2}$

   e. $\sqrt[3]{\frac{1}{8}}$

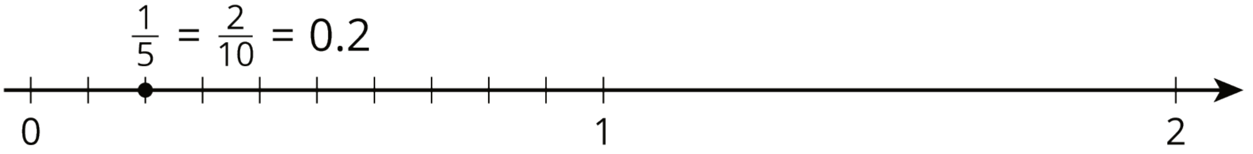
Student Response

1. Answers vary. Sample responses: $\frac{2}{10}$, -$\frac{2}{1}$, $\frac{333}{1000}$, $\frac{10}{1}$, -$\frac{1000001}{1000000}$, $\frac{1}{3}$
2. 0.375, 1.4, 0.999, 55.5, 0.5

Activity Synthesis

The purpose of this discussion is to highlight different strategies for rewriting rational numbers in different forms. Select previously identified students to share their solutions, including one student who wrote $\frac{1}{5}$ for 0.2 and another who wrote $\frac{2}{10}$. For the problems with roots, the values were purposefully chosen to emphasize to students that just because a number is written with a square or cube root does not mean it is not rational.

For 0.2, draw a number line with the numbers 0, 1, and 2 with plenty of space between the integers. Subdivide the segment from 0 to 1 into 5 equal pieces, and then ask where to plot $\frac{1}{5}$. Then label $\frac{1}{5}$ and ask how we can see that this is $\frac{2}{10}$. (Subdivide each fifth into two equal pieces—now each one is $\frac{1}{10}$.) Then label the point $\frac{2}{10}$ and 0.2.



Remind students that the point corresponds to a rational number, and we have a lot of different ways we can represent that number.

14.4 Zooming In On $\frac{2}{11}$

10 minutes

This activity continues with the work from the previous activity by examining a decimal representation of a rational number that, when written as a decimal, repeats forever. The purpose of this activity is for students to use repeated reasoning with division to justify to themselves that $0.1818...$ repeats the digits 1 and 8 forever (MP8).

Building On

- 7.NS.A.2.d

Addressing

- 8.NS.A.1

Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. Do not provide access to calculators. Tell students that now we are going to think about the decimal representation of $\frac{2}{11}$. Remind students that they should be prepared to explain their reasoning for each step in the activity.

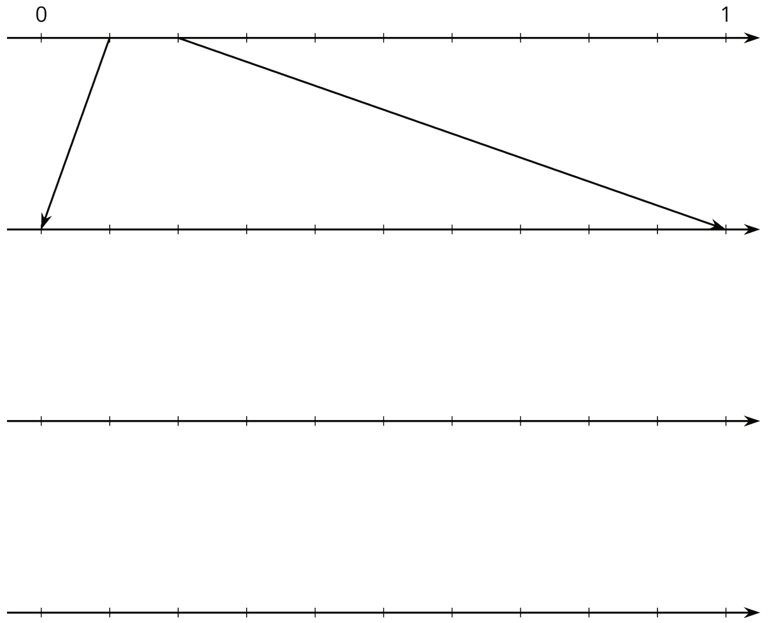
If you think students need a reminder of how the zooming number lines work, which were used in an earlier unit, demonstrate how to show where $\frac{1}{8}$ is using 3 number lines, starting with one from 0 to 1.

Students in groups of 2. Give 2 minutes for students to begin individually and then ask students to discuss their work with their partner and resolve any differences. Follow with a whole-class discussion.

Access for Students with Disabilities

Representation: Internalize Comprehension. Begin with a physical demonstration of how to use a zooming number line using a simpler fraction to support connections between new situations and prior understandings. Consider using these prompts: “What does the second number line represent?”, or “How can you relate this example to the fraction mentioned in the task statement?”
Supports accessibility for: Conceptual processing; Visual-spatial processing

Student Task Statement



1. On the topmost number line, label the tick marks. Next, find the first decimal place of $\frac{2}{11}$ using long division and estimate where $\frac{2}{11}$ should be placed on the top number line.
2. Label the tick marks of the second number line. Find the next decimal place of $\frac{2}{11}$ by continuing the long division and estimate where $\frac{2}{11}$ should be placed on the second number line. Add arrows from the second to the third number line to zoom in on the location of $\frac{2}{11}$.
3. Repeat the earlier step for the remaining number lines.
4. What do you think the decimal expansion of $\frac{2}{11}$ is?

Student Response

1. A number line from 0 to 1, each tick mark increasing by 0.1. A point for $\frac{2}{11}$ is located at 0.1.
2. A number line from 0.1 to 0.2, each tick mark increasing by 0.01. A point for $\frac{2}{11}$ is located at 0.18.
3. A number line from 0.18 to 0.19, each tick mark increasing by 0.001. A point for $\frac{2}{11}$ is located at 0.181.
4. Answers vary. Sample response: I think the decimal expansion of $\frac{2}{11}$ is $0.1818...$

Are You Ready for More?

Let $x = \frac{25}{11} = 2.272727...$ and $y = \frac{58}{33} = 1.75757575...$.

For each of the following questions, first decide whether the fraction or decimal representations of the numbers are more helpful to answer the question, and then find the answer.

- Which of x or y is closer to 2?
- Find x^2.

Student Response

- y is closer to 2. From the decimal expansion, we can see that y is less than .25 units away from 2, but x is more than .25 units away. We could also this from the fraction representation, though it seems slightly more time-consuming.
- $x^2 = \frac{625}{121}$. This is an easy calculation from the fraction representation, but would be very challenging from the decimal representation. Indeed, the decimal expansion for $\frac{625}{121}$ is $5.\overline{1652892561983471074380}$.

Activity Synthesis

The purpose of this discussion is to explicitly state the repeated reasoning and successive approximation used to calculate each digit after the decimal point for $\frac{2}{11}$. Select one or two students to share their reasoning about the decimal representation of $\frac{2}{11}$.

Tell students that we often find that the decimal representation of a rational number repeats like this, and we have a special notation to represent it. Then write

$$0.181818181818181818... = 0.\overline{18}$$

Lesson Synthesis

This lesson was about rational numbers and their decimal representations.

- “What is a rational number?” (A fraction (or its equivalent) or its opposite.)
- “What do we know about the decimal expansion of rational numbers?” (The decimal expansion always eventually *repeats*. Sometime the repeating part is zeros, like 0.250000 . . . in which case we can also say it *terminates*.)

14.5 An Unknown Rational Number

Cool Down: 5 minutes

Addressing

- 8.NS.A

Student Task Statement

Explain how you know that -3.4 is a rational number.

Student Response

Answer vary. Sample response: $-3.4 = -\frac{34}{10}$, so it is the opposite of a fraction and therefore rational.

Student Lesson Summary

We learned earlier that rational numbers are a fraction or the opposite of a fraction. For example, $\frac{3}{4}$ and $-\frac{5}{2}$ are both rational numbers. A complicated-looking numerical expression can also be a rational number as long as the value of the expression is a positive or negative fraction. For example, $\sqrt{64}$ and $-\sqrt[3]{\frac{1}{8}}$ are rational numbers because $\sqrt{64} = 8$ and $-\sqrt[3]{\frac{1}{8}} = -\frac{1}{2}$.

Rational numbers can also be written using decimal notation. Some have finite decimal expansions, like 0.75, -2.5, or -0.5. Other rational numbers have infinite decimal expansions, like 0.7434343 . . . where the 43s repeat forever. To avoid writing the **repeating** part over and over, we use the notation $0.7\overline{43}$ for this number. The bar over part of the expansion tells us the part which is to repeat forever.

A decimal expansion of a number helps us plot it accurately on a number line divided into tenths. For example, $0.7\overline{43}$ should be between 0.7 and 0.8. Each further decimal digit increases the accuracy of our plotting. For example, the number $0.7\overline{43}$ is between 0.743 and 0.744.

Glossary

- repeating decimal

Lesson 14 Practice Problems

Problem 1

Statement

Andre and Jada are discussing how to write $\frac{17}{20}$ as a decimal.

Andre says he can use long division to divide 17 by 20 to get the decimal.

Jada says she can write an equivalent fraction with a denominator of 100 by multiplying by $\frac{5}{5}$, then writing the number of hundredths as a decimal.

a. Do both of these strategies work?

b. Which strategy do you prefer? Explain your reasoning.

c. Write $\frac{17}{20}$ as a decimal. Explain or show your reasoning.

Solution

a. Yes, both strategies are effective.

b. Answers vary. Sample responses:
   - I prefer Jada's method because I can calculate it mentally.
   - I prefer Andre's method because it always works, even if the denominator is not a factor of 100.

c. 0.85. Explanations vary. Sample explanation: $\frac{17}{20} \boldsymbol{\cdot} \frac{5}{5} = \frac{85}{100}$, so $\frac{17}{20}$ equals 0.85.

Problem 2

Statement

Write each fraction as a decimal.

a. $\sqrt{\frac{9}{100}}$

b. $\frac{99}{100}$

c. $\sqrt{\frac{9}{16}}$

d. $\frac{23}{10}$

Solution

a. 0.3

b. 0.99

c. 0.75

d. 2.3

Problem 3

Statement

Write each decimal as a fraction.

a. $\sqrt{0.81}$

b. 0.0276

c. $\sqrt{0.04}$

d. 10.01

Solution

a. $\frac{9}{10}$

b. $\frac{276}{10000}$ (or equivalent)

c. $\frac{1}{5}$ (or equivalent)

d. $\frac{1001}{100}$ (or equivalent)

Problem 4

Statement

Find the positive solution to each equation. If the solution is irrational, write the solution using square root or cube root notation.

a. $x^2 = 90$

b. $p^3 = 90$

c. $z^2 = 1$

d. $y^3 = 1$

e. $w^2 = 36$

f. $h^3 = 64$

Solution

a. $x = \sqrt{90}$

b. $p = \sqrt[3]{90}$

c. $z = 1$

d. $y = 1$

e. $w = 6$

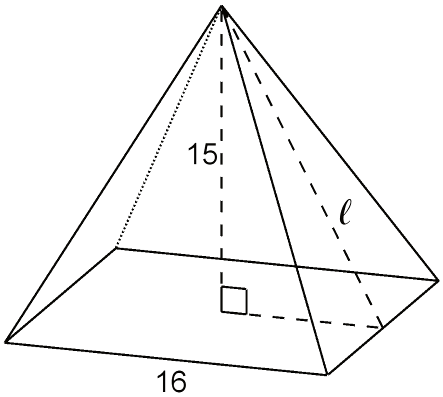
f. $h = 4$

(From Unit 8, Lesson 13.)

Problem 5

Statement

Here is a right square pyramid.



a. What is the measurement of the slant height ℓ of the triangular face of the pyramid? If you get stuck, use a cross section of the pyramid.

b. What is the surface area of the pyramid?

Solution

a. 17 units ($15^2 + 8^2 = 289$ and $\sqrt{289} = 17$)

b. 800 square units (The pyramid is made from a square and four triangles. The square's area, in square units, is $16^2 = 256$. Each triangle's area, in square units, is $\frac{1}{2} \cdot 16 \cdot 17 = 136$. The surface area, in square units, is $256 + 4 \cdot 136 = 800$.)

(From Unit 8, Lesson 10.)