Lesson 11: Stained-Glass Windows

Goals

- Apply circumference and area of circles to calculate the cost of a stained-glass window, and explain (orally and in writing) the solution method.
- Design a stained-glass window that could be built for a given dollar amount, and present (orally, in writing, and through other representations) a justification that it costs less than the limit.
- Make simplifying assumptions to solve problems about a stained-glass window.

Learning Targets

- I can apply my understanding of area and circumference of circles to solve more complicated problems.

Lesson Narrative

This culminating lesson is optional. In this lesson students work on several tasks that combine circumference and area ideas and computations. Students are given a design for a stained-glass window and the prices of the different components. They decide if it would be possible to produce the window for a certain amount of money. Students must make some assumptions about the shapes in the design and about how the different materials are sold.

The second task asks how scaling the window will affect the cost, bringing in ideas from a previous unit. Since measurements of both length and area are involved, the total cost does not simply increase by the scale factor nor by the square of the scale factor. In the last task, students invent their own design for a stained-glass window that could be produced given a cost constraint.

The series of tasks provides many opportunities to engage in different aspects of mathematical modeling (MP4) and strategic use of tools (MP5).

Alignments

Addressing

- 7.EE.B.3: Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. $
- 7.G.A.1: Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

- 7.G.B.4: Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

Instructional Routines

- Group Presentations
- MLR3: Clarify, Critique, Correct
- MLR6: Three Reads
- MLR7: Compare and Connect
- MLR8: Discussion Supports
- Think Pair Share

Required Materials

Blank paper
Compasses
Four-function calculators
Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Required Preparation

Four-function calculators are optional but recommended to take the focus off computation.

Student Learning Goals

Let’s use circumference and area to design stained-glass windows.

11.1 Cost of a Stained-Glass Window

Optional: 20 minutes
The purpose of this activity is for students to apply what they have learned about circles to solve a multi-step problem (MP1). Students find the area and perimeter of geometric figures whose boundaries are line segments and fractions of circles and use that information to calculate the cost of a project. The shape of the regions in the stained-glass window are left unspecified on purpose to give students an opportunity to engage in an important step of the mathematical modeling cycle - making simplifying assumptions (MP4). Assuming the curves in the design are arcs of a circle is not only reasonable, it is the most expedient assumption to make as well. As students work, prompt them to recognize that they are making this assumption and to make it explicit.

Another opportunity for mathematical modeling in this activity is to discuss if it is reasonable that a person only has to pay for the glass used in the final window and not for possible scraps of glass left over from cutting out the shapes. In reality, if they had to buy the glass at a store, the glass would likely come in square or rectangular sheets and they would need to buy more than they were going to use. If these issues come up, encourage students to keep note of the decisions they are making and to recognize that different choices would lead to different results.

Addressing

- 7.EE.B.3
- 7.G.B.4

Instructional Routines

- MLR6: Three Reads
- MLR8: Discussion Supports
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 5 minutes of quiet work time followed by partner and whole-class discussions.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Provide students with a graphic organizer to organize the information provided in the problem and to structure their problem-solving strategy. The graphic organizer should include the prompts: “What do I need to find out?”, “What do I need to do?”, “How I solved the problem.”, and “How I know my answer is correct.” *Supports accessibility for: Language; Organization*

Access for English Language Learners

Reading: *MLR6 Three Reads.* Use this routine to support reading comprehension of this problem, without solving it for students. In the first read, students read the problem with the goal of comprehending the situation (Students are designing a stained-glass window to hang in the school entryway.). In the second read, ask students to identify important quantities that can be counted or measured (the length and width of the window; the cost per square foot of colored glass; the cost per square foot of clear glass). In the third read, reveal the question, "Do they have enough money to cover the cost of making the window?" Ask students to brainstorm possible strategies to solve the problem. This will help students concentrate on making sense of the situation before rushing to a solution or method.
Design Principle(s): Support sense-making

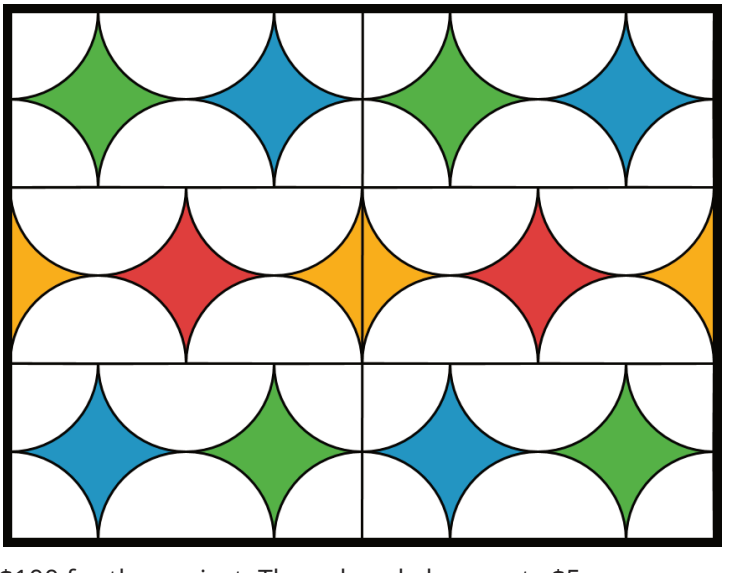
Anticipated Misconceptions

Since there are multiple steps in solving this problem, some students may benefit from having their calculations checked along the way so one early error does not impact the final result.

Some students may struggle finding the diameter or radius lengths. Encourage these students to cut one individual panel, separate the clear glass from the colored glass, and rearrange the figures to see how to determine the length of the diameter and radius.

Student Task Statement

The students in art class are designing a stained-glass window to hang in the school entryway. The window will be 3 feet tall and 4 feet wide. Here is their design.



They have raised $100 for the project. The colored glass costs $5 per square foot and the clear glass costs $2 per square foot. The material they need to join the pieces of glass together costs 10 cents per foot and the frame around the window costs $4 per foot.

Do they have enough money to cover the cost of making the window?

Student Response

Yes, they need about $93. Possible strategy:

Assume that the students only have to pay for the glass they use and not the scraps they cut away.

First, we need to find the area of the clear glass and the area of the colored glass. The entire window is 3 ft by 4 ft and has an area of $12\ \text{ft}^2$. There are 6 smaller rectangles. Each of these rectangles has a total of 2 full circles of clear glass, because $\frac{4}{2} = 2$ and $\frac{2}{2} + \frac{4}{4} = 2$. In the entire window, there are 12 complete circles of clear glass. Each circle has diameter 1 ft, radius $\frac{1}{2}$ ft, and area $\frac{1}{4}\pi\ \text{ft}^2$. The area of the clear glass is $12 \cdot \frac{1}{4}\pi$, or approximately $9.42\ \text{ft}^2$. That means the area of colored glass is approximately $12 - 9.42$, or $2.58\ \text{ft}^2$.

Next, we need to find the total length of the seams between the pieces of glass and the frame around the window. The 12 circles each have a circumference of 1π ft, which makes 12π ft or about 37.68 ft of curved seams. There are also 11 ft of straight seams, because $4 + 4 + 3 = 11$. All together there are about 48.68 ft of seams. Finally, there is 14 ft of frame all the way around the window.

Next, we can calculate how much each material will cost. The clear glass will cost $9.42 \cdot 2$, or $18.84. The colored glass will cost $2.58 \cdot 5$, or $12.90. The seams will cost $48.68 \cdot 0.10$, or $4.87. The frame will cost $14 \cdot 4$, or $56. The total cost of all the materials is about $93, because $18.84 + 12.90 + 4.87 + 56.00 = 92.61$. If these assumptions are accurate, they have just enough money to buy the materials, but if they need to pay for the scraps they cut off or if accidentally they break pieces as they go, they don't have a lot of extra money.

Activity Synthesis

As groups complete the activity, combine groups of 2 to make groups of 4. If possible, combine groups who solved the problem in different ways. Display the following questions for all to see and tell the group of 4 to discuss:

- Did you get the same answer? Why or why not?
- Did you use the same strategy? What was the same or different in your work?
- Did you make any assumptions as you worked on the problem?

Ask groups to share the similarities and differences they found in their work. Use MLR 8 (Discussion Supports) to revoice comparison statements and assumptions; ask for details and examples. After each group shares, ask the students if they had any of the same conversations in their own group so as to not have repetitive explanations. Every group does not need to share if the same conversation was had.

Ask students to make explicit any assumptions they made in their work. If it does not come up, bring out the assumption that the shapes are parts of circles and that the total cost only takes into account the exact area and lengths shown in the figure.

11.2 A Bigger Window

Optional: 10 minutes
This is a continuation of the previous one. Students use their cost computations from the previous activity to find the cost of an enlarged version of the stained-glass window, which is now scaled by a factor of 3. Students recognize that the lengths of the frame and seams will increase by a factor of 3, while the area of the glass will increase by a factor of 3^2.

If students observe that the material for the seams and the frame has width and the scale factor would need to be applied to this measurement, ask them if they can make a simplifying assumption. The width of the seams is never specified or taken into account in the calculations in the previous activity so it is appropriate to continue to put this to the side, as part of the modelling process.

As students work, monitor and select students who solved the problem in different ways to share during the whole-group discussion. If there is a student who quickly assumed they could just multiply their cost from the previous activity by 3, but then realized why they could not do that, select them to share their reasoning.

Addressing

- 7.G.A.1
- 7.G.B.4

Instructional Routines

- MLR3: Clarify, Critique, Correct
- MLR8: Discussion Supports

Launch

As students work in pairs, use MLR 3 (Clarify, Critique, Correct) with the "Critique a Partial or Flawed Response" strategy. Present students with a flawed solution method by a fictitious student. For example, “I learned that when you scale something by a factor, then you multiply things by that factor. If the people want a window three times as big, I multiply what the small window costs by three and get $279. So $450 dollars is more than enough.”

After giving students some quiet think time, ask, "Why isn't $450 enough, even though $450 is more than three times the cost of the original window?" Have students work together to come up with a suggestion to fix the flawed response and possible rules for scaling areas.

Access for Students with Disabilities

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they discuss and explain strategies to solve the problem. For example: "Let's try....", "We are trying to....", "We already know....", and "We will need to know...."
Supports accessibility for: Language; Social-emotional skills

Anticipated Misconceptions

Some students might think they can multiply the original cost by 3. Encourage them to compute the lengths and areas of the new window, or remind them that while the side lengths in scaled copies increase by the scale factor, the area increases by the square of the scale factor.

Student Task Statement

A local community member sees the school's stained-glass window and really likes the design. They ask the students to create a larger copy of the window using a scale factor of 3. Would $450 be enough to buy the materials for the larger window? Explain or show your reasoning.

Student Response

No, $450 is not enough money. They would need about $468.27. The lengths of the seams and the frame are one-dimensional, so they scale by 3. The areas of the clear glass and the colored glass are two-dimensional, so they scale by 9.

$$18.84 \cdot 9 + 12.90 \cdot 9 + 4.87 \cdot 3 + 56.00 \cdot 3 = 468.27$$

Activity Synthesis

Ask selected students to share their reasoning. If there are students who still think $450 is enough money, ask them to share their reasoning. Discuss why you cannot just multiply the price of the original design by 3 to find the price of the scaled stained-glass window.

Access for English Language Learners

Speaking: MLR8 Discussion Supports. As students share how they calculated the total cost of the larger window, press for details in students' explanations by asking how they know that the areas of the clear glass and the colored glass are scaled by 9. Also, ask students how they know the lengths of the seams and the frame are scaled by 3. Listen for and amplify comments that clarify how the scale factor affects the lengths and areas of shapes. For example, the lengths are scaled by the scale factor; however, the area is scaled by the square of the scale factor because the length and width are both scaled by the same scale factor. If necessary, draw a rectangle with a length of 4 units, width of 3 units, and area of 12 square units. Then draw a scaled rectangle by a scale factor of 3 with a length of 12 units and width of 9 units. The area of the scaled rectangle is 108 square units, which is the area of the original rectangle scaled by 9 or 3^2. This will support rich and inclusive discussion about how the scale factor affects the lengths and area of shapes.
Design Principle(s): Support sense-making

11.3 Invent Your Own Design

Optional: 15 minutes
The purpose of this activity is for students to create their own stained-glass design for a given amount of money. The activity is purposefully left open to allow students to either tweak the previous design or create something completely new.

As students work, monitor and select students who either tweaked a previous design or created a new, interesting design to share during the whole-group discussion.

Addressing

- 7.G.B.4

Instructional Routines

- Group Presentations
- MLR7: Compare and Connect

Launch

Students in same groups. Remind students to include whole or partial circles in their designs.

Anticipated Misconceptions

Some students may think they need to create a new design and struggle getting started. Point these students to the designs in previous activities and ask how they could modify these designs to meet the cost requirement.

Student Task Statement

Draw a stained-glass window design that could be made for less than $450. Show your thinking. Organize your work so it can be followed by others.

Student Response

Answers vary

Activity Synthesis

Display students' designs for all to see and ask students to explain how they knew their design met the cost requirement. Allow other students to ask questions of the student who is sharing their design.

Access for English Language Learners

Speaking, Listening: MLR7 Compare and Connect. As students prepare their stained-glass window designs, look for students with different solution methods. As students investigate each other's work, ask students to share what worked well in a particular approach. During this discussion, listen for and amplify any comments about what might make the calculation of the cost more precise. Then encourage students to make connections between the quantities used to calculate the cost of the glass and the circles in the stained-glass window design. During this discussion, listen for and amplify language students use to interpret quantities as the total area of the whole or partial circles in their design. This will foster students' meta-awareness and support constructive conversations as they compare strategies for calculating the cost of a stained-glass window and make connections between quantities and the area of circles.
Design Principle(s): Cultivate conversation; Maximize meta-awareness