Lesson 6: Fermi Problems

Goals

- Calculate a rough estimate for quantities that are difficult or impossible to measure directly and explain (orally) the reasoning.
- Choose an appropriate level of accuracy when reporting estimates of quantities.
- Make simplifying assumptions to solve problems about estimating quantities.

Lesson Narrative

This lesson is optional. The activities in this lesson plan are sometimes called “Fermi problems” after the famous physicist Enrico Fermi. A Fermi problem requires students to make a rough estimate for quantities that are difficult or impossible to measure directly. Often, they use rates and require several calculations with fractions and decimals, making them well-aligned to grade 7 work. Fermi problems are examples of mathematical modeling (MP4), because one must make simplifying assumptions, estimates, research, and decisions about which quantities are important and what mathematics to use. They also encourage students to attend to precision (MP6), because one must think carefully about how to appropriately report estimates and choose words carefully to describe the quantities.

In determining your exact age, the level of accuracy depends on how exact you know your moment of birth: to the day? the minute? the second? In determining the number of heartbeats in your lifetime, it is impossible to know the exact answer because we do not have access to all of the necessary information (and even if we did, the numbers involved are too large to count). In determining the number of hairs on your head, there is no method available other than counting, yet there are no tools to do this accurately. These two scenarios are far more difficult to estimate with the same degree of accuracy as your age. You can reliably determine your age within a day: it would be very difficult to estimate heartbeats or hairs with this level of accuracy.

Any of these tasks can stand on its own. Choose those that you have time for. It is likely that it would take more than a single day to do all of the tasks in this lesson. Make sure to leave plenty of time for discussion. Important topics of discussion should include *why* the quantities in question are difficult to measure and the level of precision we should use to record our estimates.

As with all lessons in this unit, all related standards have been addressed in prior units. This lesson provides an *optional* opportunity to go deeper and make connections between domains.

Alignments

Building On

- 4.MD.A.2: Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

Addressing

- 7.G.B: Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.
- 7.NS.A.3: Solve real-world and mathematical problems involving the four operations with rational numbers.Computations with rational numbers extend the rules for manipulating fractions to complex fractions.
- 7.RP.A: Analyze proportional relationships and use them to solve real-world and mathematical problems.
- 7.RP.A.3: Use proportional relationships to solve multistep ratio and percent problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.

Instructional Routines

- MLR2: Collect and Display
- MLR8: Discussion Supports
- Poll the Class

Required Materials

Four-function calculators
Measuring tapes
Stopwatches
String

Required Preparation

During the A Heart Stoppingly Large Number activity, students have the option of measuring one another's pulse rate. If measuring pulse rate, students will need access to stopwatches.

During the All Hairs on Your Head activity, students have the option of measuring their head. If measuring heads, students will need access to string and measuring tape marked in centimeters.

Student Learning Goals

Let's estimate some quantities.

6.1 How Old Are You?

Optional: 20 minutes
Students attempt to calculate their *exact* age. Because this is a constantly changing quantity, they need to think carefully about how accurately to report the answer. The mathematics involved is multiple unit conversions in the context of time. In addition to the fact that our age is always growing, we may not know with great accuracy when we were born.

Building On

- 4.MD.A.2

Addressing

- 7.NS.A.3
- 7.RP.A

Instructional Routines

- MLR8: Discussion Supports

Launch

Arrange students into groups of 4. Ask them to order themselves according to their age: who is the youngest? Who is the oldest? What information do you need to decide?

Tell students that they will be finding their *exact* age.

Provide access to calculators. Give students 10 minutes of group work time, followed by whole-class discussion.

Anticipated Misconceptions

Students may struggle with what is meant by *at this moment* in the prompt. Ask them how they interpret this: the question is intended to be flexible so students can interpret *at this moment* as being when they read the question, or the day of class, or in another way.

Student Task Statement

What is your *exact* age at this moment?

Student Response

Answers vary. Sample solution: I was born on April 4, 2003 at 12:09 p.m. Today is December 17, 2016, and right now it is 10:55 a.m. So I am 13 years, 256 days, 22 hours, and 46 minutes old right now. I can’t get any more exact because I don’t know the second I was born, and even then I wouldn’t know the fraction of a second. Plus, I’m not sure if I should answer the question for when I begin to write my solution, or for when I will be done writing my solution, because I keep getting older.

Activity Synthesis

Invite students to share answers and discuss difficulties in answering the question. The discussion should include the following points:

- We can give an estimate, but the question cannot be answered because we probably do not know the exact time when we were born and “at this moment” keeps moving forward.

- How we express our answer depends on what we know about the time we were born. We probably know the day, but do we know the time? Should we answer to the nearest hour? Minute? Second?
- Did you take into account leap years?

Ask students how they usually answer the question, "How old are you?" (This will probably be in whole number of years lived.) Why? (Because this whole number communicates enough information for most purposes.) After more than six months have passed since someone's last birthday, the person still doesn't normally round up when they report their age, even though this is customary for reporting many other types of measurements.

Access for English Language Learners

Speaking: MLR8 Discussion Supports. Use this routine to support whole-class discussion. For each response or observation that is shared, ask students to restate and/or revoice what they heard using precise mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to produce language as they interpret the reasoning of others.
Design Principle(s): Support sense-making

6.2 A Heart Stoppingly Large Number

Optional: 15 minutes
This activity focuses on another complex calculation: the number of times our heart has beaten in our lifetime. In addition to all of the precision issues in determining how old we are from the previous activity, there is an additional level of complexity as our heart does *not* beat at a constant rate. For this situation, we need to make a reasoned estimate, but there is no hope of getting an exact answer. This makes our lack of knowledge of the exact time of our birth (or the exact time when we are solving the problem) unimportant because a few minutes or hours will not significantly impact the answer.

Addressing

- 7.RP.A.3

Instructional Routines

- MLR2: Collect and Display
- Poll the Class

Launch

Arrange students in groups of 2. Tell students they will be investigating how many times their heart has beaten in their lifetime. Ask students to make an estimate and then poll the class. One way to accomplish this is to display a table like this and complete it with the number of students whose estimate is in each range.

estimate for number of heartbeats	number of students
around 10,000	
around 100,000	
around 1,000,000	
around 10,000,000	
around 100,000,000	
around 1,000,000,000	
around 10,000,000,000	
around 100,000,000,000	

Ask students to brainstorm what they need to know or investigate to answer this question. Provide access to stopwatches (if students will estimate their own heartbeats) and calculators. Alternatively, share with them that a normal heart beat range is 60 to 100 beats per minute with higher rates when we exercise and sometimes lower rates at rest.

Give students 10 minutes of partner work time followed by a whole-class discussion.

Access for English Language Learners

Conversing: MLR2 Collect and Display. While students work in their groups to measure heart rates and make calculations, circulate and listen to the language students use related to their estimation process (e.g., “rate,” “per,” “constant,” etc.). Record their language on a visual display and update it throughout the remainder of the lesson. Remind students to borrow language from the display as needed. This helps students produce mathematical language as they justify their estimations.
Design Principle(s): Optimize output (for justification); Cultivate conversation

Anticipated Misconceptions

Some students may struggle with simplifying the assumptions needed to solve this problem. Encourage these students by:

- Telling them to think about how old they are if they have done the previous activity.
- Telling them the normal rate of heartbeats per minute.
- Asking them how they can find out about how many times a heart beats in an hour (multiply minute rate by 60), in a day (multiply hourly rate by 24), in a year, and so on.

Student Task Statement

How many times has your heart beat in your lifetime?

Student Response

Answers vary. Sample solution: I am about 13 years old. There are 365 days per year, 24 hours per day, and 60 minutes per hour. So that is $13 \cdot 365 \cdot 24 \cdot 60$, which is about 6.8 million minutes. I just took my pulse, and my heart rate is about 75 beats per minute. Even though that changes over time, I'll use it to estimate the total number of times my heart has beaten in my lifetime so far by multiplying the number of minutes I've been alive by 75. That gives about 500 million heartbeats.

Activity Synthesis

Invite students to share answers and discuss the difficulties they encountered answering these questions:

- "How did you estimate how often your heart beats?" (By checking pulse, looking online, etc.)
- "Is your heart rate always the same?" (No, it is faster when I exercise and slower when I sleep.)
- "Did you use the information you found for your age?" (No, because I only have an estimate for my pulse *or* yes because I already had the information available.)

Time permitting, students can be encouraged to check their pulse while resting and then after a short amount of exercise (jumping jacks, running in place, push ups). This gives them an idea of the variability involved in how frequently our heart beats.

The calculation of our number of heartbeats is only an estimate. One way we can indicate this is with the way we report the final answer. An answer of 524,344,566 would not be appropriate because that makes it look like it is an exact answer. An answer of 500,000,000 makes it clear that we are only making an estimate. Alternatively, students might say that the number of heartbeats is between 400,000 and 700,000 with the estimate of 400,000 coming from the low value of 60 beats per minute and the estimate of 700,000 coming from the high value of 100 beats per minute.

6.3 All the Hairs on Your Head

Optional: 20 minutes

Students estimate how many hairs they have on their head. Although there is a definite number of hairs on each student's head, finding this number exactly is not feasible: although it is a much smaller number than the number of heartbeats in a student's lifetime, it is still too large to count and, even if we could count, we would need to cut all of those hairs to do so! An additional geometric layer of estimation comes into play in this task as students need to estimate the surface area of part of their head. If one or more students in the class are bald or are otherwise sensitive about their hair, consider asking them to estimate the number of hairs on another student's head.

Addressing

- 7.G.B
- 7.RP.A.3

Instructional Routines

- Poll the Class

Launch

Arrange students in groups of 2. Provide access to calculators. Before starting, poll the class to guess how many hairs they have on their head. One way to accomplish this is to display a table like this, and record the number of students with each guess:

number of hairs	number of guesses
about 100	
about 1,000	
about 10,000	
about 100,000	
about 1,000,000	

Next, ask students to brainstorm the information they need to answer the question. Provide the following information below when students ask for it.

Data to keep in reserve and share with students as needed:

- The number of hairs per square cm varies from person to person, but for this analysis, we can assume approximately 150 hairs per square cm.
- Students have to be creative in their estimates for the area of their scalp. One possibility is to approximate it with a circle. 600 square centimeters is a good estimate, but students may come up with slightly different estimates.

Provide access to string, tape measures, and calculators. After taking 5 minutes to introduce the activity, give students 10 minutes of partner work time, followed by whole-class discussion.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Provide students with a graphic organizer for recording measurements and calculations of areas of circles to represent hair on head. *Supports accessibility for: Language; Organization*

Anticipated Misconceptions

Since the scalp is three dimensional, students may not think of approximating the scalp with a circle and may struggle with trying to identify the size of the circle. Consider asking these students:

- What two-dimensional shape can you use to model the part of your scalp covered with hair? (A circle)
- How can you estimate the diameter of the circle? (Measuring from front to back or from side to side)
- How do you estimate the area of the circle using the diameter? (The diameter gives the radius, and then the area can be found once the radius is known.)

Student Task Statement

How many strands of hair do you have on your head?

Student Response

Answers vary. Sample response: a piece of string around the head, from the front of the forehead to the back of the neck measures about 28 cm. A piece of string going from ear to ear also measures about 28 cm. So it is reasonable to model the scalp with a circle whose diameter is 28 cm. The radius is 14 cm, and the area of the circle is $\pi \boldsymbol{\cdot} 14^2 \approx 600$ square cm. If there are 150 strands of hair per square cm, that means that there are about 90,000 strands of hair on the head.

Activity Synthesis

Invite students to share their answers and methods. Consider asking the following questions:

- “Why is it challenging to find the exact number of hairs on your head?” (There are too many to count, and they are too small to count accurately. The number changes as time goes by.)
- “How can you estimate the number of hairs on your head?” (Estimate the area of the part of the head covered by hair. Try to find how many hairs there are on a small part of the head.)
- “How did you estimate the area of your head?” (Approximated area with a circle. Covered the head with little pieces of paper (sticky notes) and added up their areas.)

This is a situation where there is a definite number of hairs on our head, but because they are so small (and because they are on our head), we cannot calculate this number exactly. Fortunately, for almost any purpose, an estimate will do.