Lesson 3: Tessellating Polygons

Goals

- Generalize (orally) that any triangle or quadrilateral can be used to tessellate the plane.

Lesson Narrative

In this third in the sequence of three lessons, students examine tessellations using non-regular polygons. Students show that *any* triangle can be used to tessellate the plane and similarly for any quadrilateral. Pentagons do not work in general, for example, a regular pentagon cannot be used to tessellate the plane.

Tessellating the plane with a triangle uses the important idea, studied in the sixth grade, that two copies of a triangle can be put together to make a parallelogram. Tessellating the plane with a quadrilateral uses rigid motions of the plane and the fact that the sum of the angles in a quadrilateral is always 360. One example of a plane tessellation with a special pentagon also uses rotations.

Alignments

Addressing

- 8.G.A: Understand congruence and similarity using physical models, transparencies, or geometry software.

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR7: Compare and Connect
- MLR8: Discussion Supports
- Poll the Class

Required Materials

Tracing paper
Bundles of "patty paper" are available commercially for a very low cost. These are small sheets (about 5" by 5") of transparent paper.

Required Preparation

If students are using the applets in the digital versions of the activities, tracing paper may not be needed.

Student Learning Goals

Let’s make tessellations with different polygons.

3.1 Triangle Tessellations

Optional: 15 minutes (there is a digital version of this activity)

In this activity, students experiment with copies of a triangle (no longer equilateral) and discover that it is always possible to build a tessellation of the plane. A key in finding a tessellation with copies of a triangle is to experiment with organizing copies of the triangle, and then reasoning that two copies of a triangle can always be arranged to form a parallelogram. Students may not remember this construction from the sixth grade, but with copies of the triangle to experiment with, they will find the parallelogram or a different method. These parallelograms can then be put together in an infinite row, and these rows can then be stacked upon one another to tessellate the plane.

Addressing

- 8.G.A

Instructional Routines

- MLR7: Compare and Connect

Launch

Assign different triangles to different students or groups of students. Provide access to tracing paper if using the print materials. If using the digital materials, the activity can be done in the applet.

If students finish early, consider asking them to work on building a different tessellation or coloring their tessellation.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. To support development of organizational skills, check in with students or groups of students within the first 2-3 minutes of work time. Look for students who are using two copies of the triangle to form a parallelogram in order to create a tessellation.
Supports accessibility for: Memory; Organization

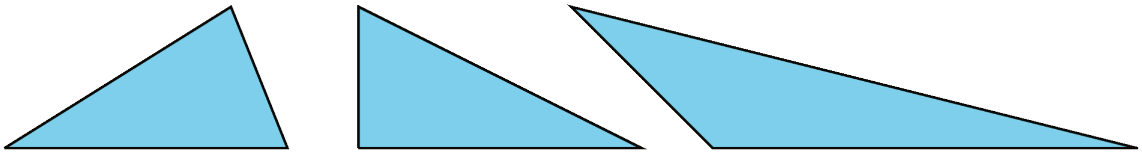
Anticipated Misconceptions

Students may struggle to put together copies of their triangle in a way that can be continued to tessellate the plane.

- Ask these students to put together two copies of the triangle.
- If they have made a parallelogram, ask them what kind of quadrilateral they have made.
- If they have not made a parallelogram, ask them if there is a different way they can combine two copies of the triangle.

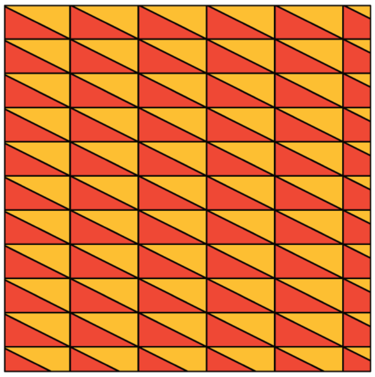
Student Task Statement

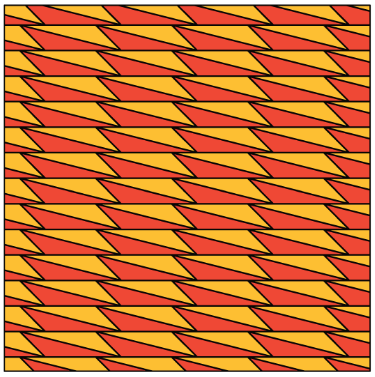
Your teacher will assign you one of the three triangles. You can use the picture to draw copies of the triangle on tracing paper. Your goal is to find a tessellation of the plane with copies of the triangle.

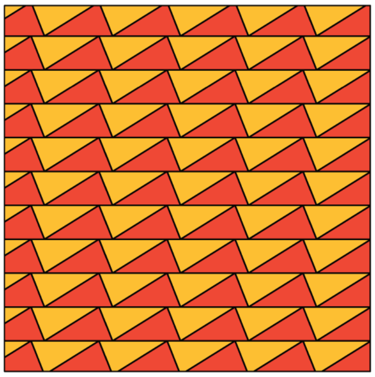


Student Response

Sample response for each type of triangle:







Activity Synthesis

Invite several students to share their tessellations for all to see.

Consider asking the following questions to help summarize the lesson:

- “Were you able to make a tessellation with copies of your triangle?” (Most students should respond yes.)
- “How did you know that you could continue your pattern indefinitely to make a tessellation?” (Any parallelogram can be used to tessellate the plane as they can be placed side by side to make infinite “rows” or “columns” and then these rows or columns can be displaced to fill up the plane.)

Share some of the tessellation ideas students come up with and relate them back to previous work, that is the tessellation of the plane with rectangles and parallelograms.

Access for English Language Learners

Representing, Conversing: MLR7 Compare and Connect. After students complete their triangle tessellations, display the work around the room. Invite students to tour the room to observe and compare the displays. As students circulate, ask, “Are there any drawings that are similar in appearance?” and “Did any two drawings use the same kind of triangle, but create a different tessellation pattern?” Finish by inviting students to share the similarities and differences they discovered with the class. Look for language identifying that “two of the same triangles” placed together create a “parallelogram” no matter the type of triangle chosen. Also, amplify language that concludes that the “plane” can be filled by “repeated triangle” patterns. This helps students draw comparisons between various triangle tessellations and explain their reasoning about how they can fill the plane.
Design Principle(s): Cultivate conversation; Maximize meta-awareness

3.2 Quadrilateral Tessellations

Optional: 20 minutes (there is a digital version of this activity)
The previous activity showed how to make a tessellation with copies of a triangle. A natural question is whether or not it is possible to tessellate the plane with copies of a single quadrilateral. Students have already investigated this question for some special quadrilaterals (squares, rhombuses, regular trapezoids), but what about for an arbitrary quadrilateral? This activity gives a positive answer to this question. Pentagons are then investigated in the next lesson, and there we will find that some pentagons can tessellate the plane while others can not.

In order to show that the plane can be tessellated with copies of a quadrilateral, students will experiment with rigid motions and copies of a quadrilateral.

This activity can be made more open ended by presenting students with a polygon and asking them if it is possible to tessellate the plane with copies of the polygon.

Addressing

- 8.G.A

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- Poll the Class

Launch

Begin the activity with, “Any triangle can be used to tile the plane (some of them in many ways). Do some quadrilaterals tessellate the plane?” (Yes: squares, rectangles, rhombuses, and parallelograms.) Next, ask “Can any quadrilateral be used to tessellate the plane?” Give students a moment to ponder, and then poll the class for the number of yes and no responses. Record the responses for all to see. This question will be revisited in the Activity Synthesis.

Provide access to tracing paper.

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge of quadrilaterals, as well as 180 degree rotations around the midpoint of a side. Consider demonstrating how to use tracing paper to rotate a figure 180 degrees around the midpoint of a side. Be sure to model accuracy and precision.
Supports accessibility for: Visual-spatial processing; Organization

Access for English Language Learners

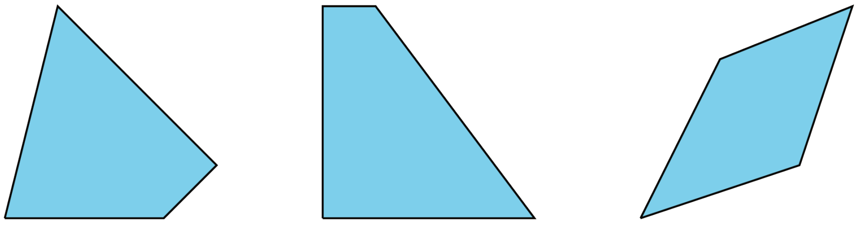
Representing, Speaking, Writing: MLR1 Stronger and Clearer Each Time. Give students a few minutes to complete the first problem. Students then turn to their partner and share their tessellation and explanation. Listeners should press for details in their explanation asking whether there are “any spaces left” and whether they “can prove that the trapezoid fits the whole plane”. Students will explain and show their tessellation to at least one more partner, each time adding detail to their explanation. The focus should be on explaining, even if they were not successful. Finish with students rewriting their initial explanation, adding any detail or corrections needed. This helps students explain their method for moving the trapezoid and use geometric language to prove whether it works.
Design Principle(s): Optimize output (for explanation); Maximize meta-awareness

Anticipated Misconceptions

Students may need to be reminded of the definition of a trapezoid: one pair of sides are contained in parallel lines.

If the figures are not traced accurately, it may be difficult to determine if the pattern, using 180-degree rotations, can be continued. Ask these students what they know about the sum of the three angles in a triangle and in a quadrilateral.

Student Task Statement



1. Can you make a tessellation of the plane with copies of the trapezoid? Explain.

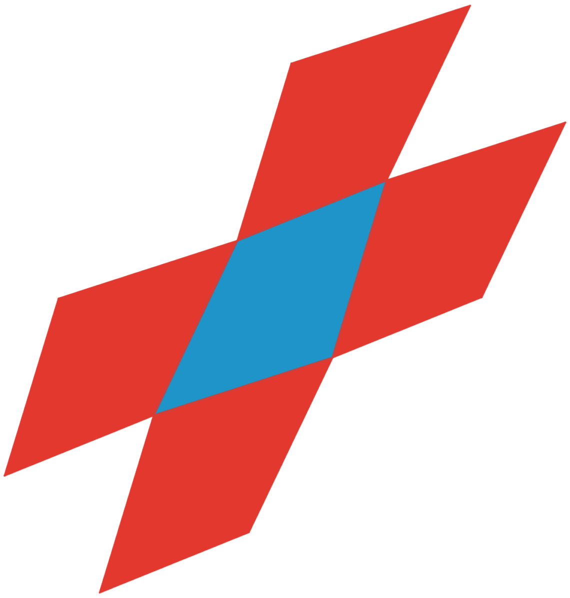
2. Choose and trace a copy of one of the other two quadrilaterals. Next, trace images of the quadrilateral rotated 180 degrees around the midpoint of each side. What do you notice?

3. Can you make a tessellation of the plane with copies of the quadrilateral from the previous problem? Explain your reasoning.

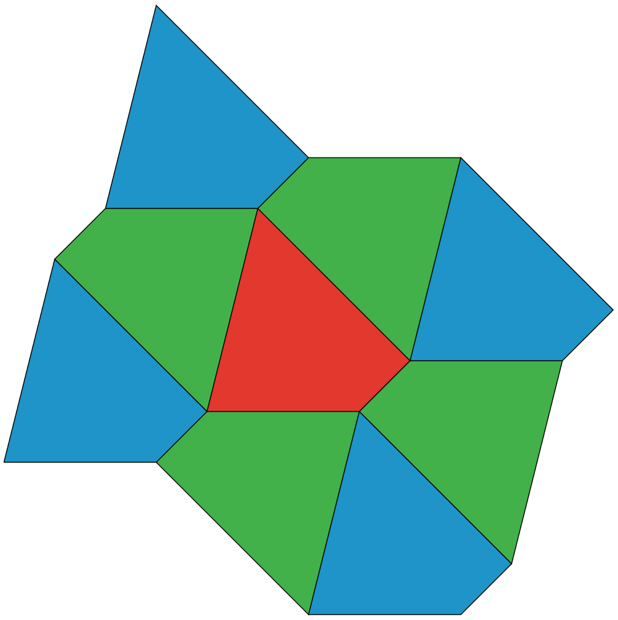
Student Response

1. Yes, rotating the trapezoid 180 degrees about the slanted side to the base makes a rectangle. The plane can be tessellated with copies of this rectangle. Rotating the trapezoid 180 degrees about the other side to the base also makes a parallelogram, and then the plane can be tessellated with copies of the parallelogram.

2. It looks like there is room to fit copies of the quadrilateral in each of the wedges at the corners. When they are filled in, it will look like a checkerboard.



3. Yes, continuing these 180-degree rotations about midpoints of sides of the quadrilateral fills in the whole plane. At each vertex, the four angles of the quadrilateral meet to make a full 360-degree circle.



Activity Synthesis

Invite some students to share their tessellations.

Discussion questions include:

- “Were you able to tessellate the plane with copies of the trapezoid?” (Yes, two of them can be out together to make a parallelogram, and the plane can be tessellated with copies of this parallelogram.)
- “What did you notice about the quadrilateral and the 180-degree rotations?” (They fit together with no gaps and no overlaps and leave space for four more quadrilaterals.)

- “How do you know that there are no overlaps?” (The sum of the angles in a quadrilateral is 360 degrees. At each vertex in the tessellation, copies of the four angles of the quadrilateral come together.)

Revisit the question from the start of the activity, “Can any quadrilateral be used to tessellate the plane?” Invite students to share if their answer has changed and explain their reasoning.

3.3 Pentagonal Tessellations

Optional: 20 minutes (there is a digital version of this activity)
All triangles and all quadrilaterals give tessellations of the plane. For the quadrilaterals, this was complicated and depended on the fact that the sum of the angles in a quadrilateral is 360 degrees. Regular pentagons that do not tessellate the plane have been seen in earlier activities. The goal of this activity is to study some types of pentagons that *do* tessellate the plane. Students make use of structure (MP7) when they relate the pentagons in this activity to the hexagonal tessellation of the plane, which they have seen earlier.

This activity can be made more open ended by presenting students with a polygon and asking them if it is possible to tessellate the plane with copies of the polygon.

Addressing

- 8.G.A

Instructional Routines

- MLR8: Discussion Supports

Launch

Ask students:

- “Can you tessellate the plane with regular pentagons?” (No.)
- “Can you think of a type of pentagon that could be used to tessellate the plane?” (A square base with a 45-45-90 triangle on top, for example.)

Arrange students in groups of two.

Print version: Provide access to tracing paper.

Access for Students with Disabilities

Action and Expression: Develop Expression and Communication. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their ideas. For example, "First, I _____ because . . .", "At the central vertex, I noticed that . . .", "I agree/disagree because . . .", or "What other details are important?"
Supports accessibility for: Language; Organization

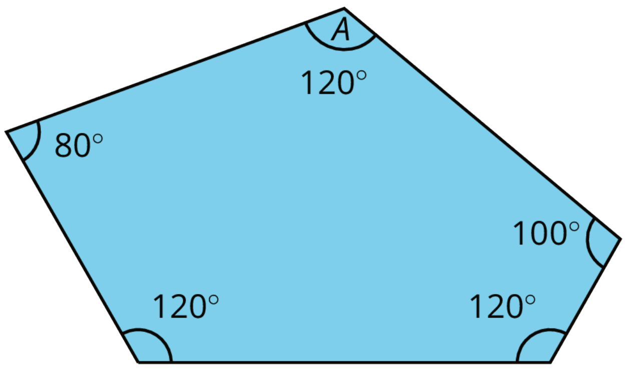
Anticipated Misconceptions

Students may struggle tracing the rotated hexagon. Ask them what happens to the segments making angle A when the hexagon is rotated about A by 120 (or 240) degrees.

Students may wonder why the hexagon that they make by putting three pentagons together is a regular hexagon. Invite these students to calculate the angles of the hexagon.

Student Task Statement

1. Can you tessellate the plane with copies of the pentagon? Explain why or why not. Note that the two sides making angle A are congruent.



   Pause your work here.

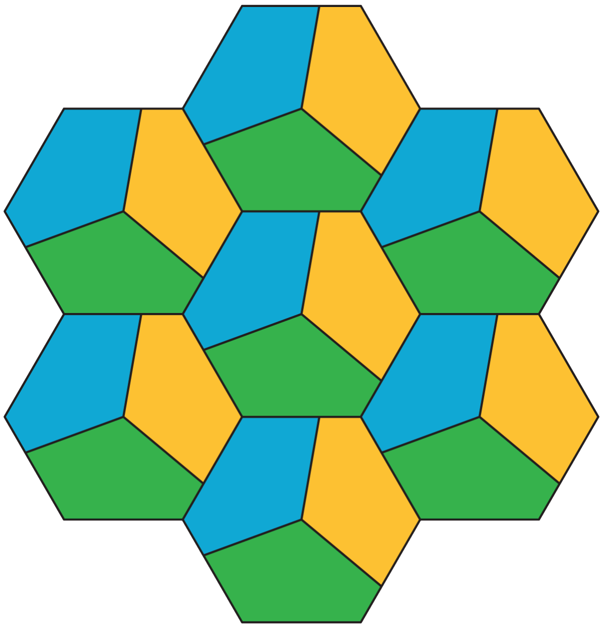
2. Take one pentagon and rotate it 120 degrees clockwise about the vertex at angle A, and trace the new pentagon. Next, rotate the pentagon 240 degrees clockwise about the vertex at angle A, and trace the new pentagon.

3. Explain why the three pentagons make a full circle at the central vertex.

4. Explain why the shape that the three pentagons make is a hexagon (that is, the sides that look like they are straight really are straight).

Student Response

1. Yes, I can make a hexagon from three pentagons and can then tessellate the plane with those hexagons.
2. See picture for item 4 showing several of the hexagons made by putting together three pentagons.
3. The angles that are joined at the central vertex each measure 120 degrees, and there are 360 degrees in the full circle.
4. Three of the six sides of the hexagon are corresponding sides of the pentagon after a rotation. For the other three sides, the angles 100 and 80 are supplementary, and so the two segments making those sides are colinear. All angles of the hexagon measure 120 degrees, so it is a regular hexagon.



Activity Synthesis

Students may be successful in building a tessellation in the first question. The following questions guide them through a method while also asking for mathematical justification. Students who are successful in the first question can verify that their tessellation uses the strategy indicated in the following, and they will still need to answer the last two questions.

Invite some students to share their tessellations.

Some questions to discuss include:

- “Does the hexagon made by three copies of the pentagon tessellate the plane?” (Yes.)

- “How do you know?” (I checked experimentally, or I noticed that all of the angles in the hexagon are 120 degrees.)
- “Why was it important that the two sides of the pentagons making the 120 angles are congruent?” (So that when I rotate my pentagon, those two sides match up with each other perfectly.)
- “What is special about this pentagon?” (Two sides are congruent, three angles measure 120 degrees. . .)

Access for English Language Learners

Representing, Conversing: MLR8 Discussion Supports. Give students a structured opportunity to revise and refine their response to the first question. After students have completed their tessellations using the hexagon, ask students to meet with 2–3 other groups for feedback. Display the questions included in the synthesis on the screen and invite each partnership to take turns asking the questions. Encourage students to press each other for detailed explanations using mathematical language. Peer questioning increases and improves student output for explanations.
Design Principle(s): Optimize output (for explanation); Cultivate conversation