Lesson 21: Sums and Products of Rational and Irrational Numbers

- Let's make convincing arguments about why the sums and products of rational and irrational numbers always produce certain kinds of numbers.

21.1: Operations on Integers

Here are some examples of integers (positive or negative whole numbers):

-25 -10 -2 -1 0 5 9 40

1. Experiment with adding any two numbers from the list (or other integers of your choice). Try to find one or more examples of two integers that:

    a. add up to another integer

    b. add up to a number that is *not* an integer

2. Experiment with multiplying any two numbers from the list (or other integers of your choice). Try to find one or more examples of two integers that:

    a. multiply to make another integer

    b. multiply to make a number that is *not* an integer

21.2: Sums and Products of Rational Numbers

1. Here are a few examples of adding two rational numbers. Is each sum a rational number? Be prepared to explain how you know.

    a. $4 + 0.175 = 4.175$

    b. $\frac{1}{2} + \frac{4}{5} = \frac{5}{10} + \frac{8}{10} = \frac{13}{10}$

    c. $-0.75 + \frac{14}{8} = \frac{-6}{8} + \frac{14}{8} = \frac{8}{8} = 1$

    d. a is an integer: $\frac{2}{3} + \frac{a}{15} = \frac{10}{15} + \frac{a}{15} = \frac{10+a}{15}$

2. Here is a way to explain why the sum of two rational numbers is rational.

   Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are fractions. That means that $a, b, c,$ and d are integers, and b and d are not 0.

   a. Find the sum of $\frac{a}{b}$ and $\frac{c}{d}$. Show your reasoning.

   b. In the sum, are the numerator and the denominator integers? How do you know?

   c. Use your responses to explain why the sum of $\frac{a}{b} + \frac{c}{d}$ is a rational number.

3. Use the same reasoning as in the previous question to explain why the product of two rational numbers, $\frac{a}{b} \boldsymbol{\cdot} \frac{c}{d}$, must be rational.

Are you ready for more?

Consider numbers that are of the form $a + b\sqrt{5}$, where a and b are whole numbers. Let's call such numbers *quintegers*.

Here are some examples of quintegers:

- $3 + 4\sqrt{5}$ $(a = 3, b = 4)$
- $7 - 2\sqrt{5}$ $(a = 7, b = \text{-}2)$
- $\text{-}5 + \sqrt{5}$ $(a = \text{-}5, b = 1)$
- 3 $(a = 3, b = 0)$.

1. When we add two quintegers, will we always get another quinteger? Either prove this, or find two quintegers whose sum is not a quinteger.

2. When we multiply two quintegers, will we always get another quinteger? Either prove this, or find two quintegers whose product is not a quinteger.

21.3: Sums and Products of Rational and Irrational Numbers

1. Here is a way to explain why $\sqrt{2} + \frac{1}{9}$ is irrational.

    - Let s be the sum of $\sqrt{2}$ and $\frac{1}{9}$, or $s = \sqrt{2} + \frac{1}{9}$.
    - Suppose s is rational.

    a. Would $s + -\frac{1}{9}$ be rational or irrational? Explain how you know.

    b. Evaluate $s + -\frac{1}{9}$. Is the sum rational or irrational?

    c. Use your responses so far to explain why s cannot be a rational number, and therefore $\sqrt{2} + \frac{1}{9}$ cannot be rational.

2. Use the same reasoning as in the earlier question to explain why $\sqrt{2} \cdot \frac{1}{9}$ is irrational.

21.4: Equations with Different Kinds of Solutions

1. Consider the equation $4x^2 + bx + 9 = 0$. Find a value of b so that the equation has:

    a. 2 rational solutions

    b. 2 irrational solutions

    c. 1 solution

    d. no solutions

2. Describe all the values of b that produce 2, 1, and no solutions.

3. Write a new quadratic equation with each type of solution. Be prepared to explain how you know that your equation has the specified type and number of solutions.

    a. no solutions

    b. 2 irrational solutions

    c. 2 rational solutions

    d. 1 solution

Lesson 21 Summary

We know that quadratic equations can have rational solutions or irrational solutions. For example, the solutions to $(x+3)(x-1)=0$ are -3 and 1, which are rational. The solutions to $x^2-8=0$ are $\pm\sqrt{8}$, which are irrational.

Sometimes solutions to equations combine two numbers by addition or multiplication—for example, $\pm 4\sqrt{3}$ and $1+\sqrt{12}$. What kind of number are these expressions?

When we add or multiply two rational numbers, is the result rational or irrational?

- The sum of two rational numbers is rational. Here is one way to explain why it is true:
    - Any two rational numbers can be written $\frac{a}{b}$ and $\frac{c}{d}$, where $a, b, c, \text{ and } d$ are integers, and b and d are not zero.
    - The sum of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ad+bc}{bd}$. The denominator is not zero because neither b nor d is zero.
    - Multiplying or adding two integers always gives an integer, so we know that ad, bc, bd and $ad+bc$ are all integers.
    - If the numerator and denominator of $\frac{ad+bc}{bd}$ are integers, then the number is a fraction, which is rational.
- The product of two rational numbers is rational. We can show why in a similar way:
    - For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, where $a, b, c, \text{ and } d$ are integers, and b and d are not zero, the product is $\frac{ac}{bd}$.
    - Multiplying two integers always results in an integer, so both ac and bd are integers, so $\frac{ac}{bd}$ is a rational number.

What about two irrational numbers?

- The sum of two irrational numbers could be either rational or irrational. We can show this through examples:
    - $\sqrt{3}$ and -$\sqrt{3}$ are each irrational, but their sum is 0, which is rational.
    - $\sqrt{3}$ and $\sqrt{5}$ are each irrational, and their sum is irrational.

- The product of two irrational numbers could be either rational or irrational. We can show this through examples:
    - $\sqrt{2}$ and $\sqrt{8}$ are each irrational, but their product is $\sqrt{16}$ or 4, which is rational.
    - $\sqrt{2}$ and $\sqrt{7}$ are each irrational, and their product is $\sqrt{14}$, which is not a perfect square and is therefore irrational.

What about a rational number and an irrational number?

- The sum of a rational number and an irrational number is irrational. To explain why requires a slightly different argument:
    - Let R be a rational number and I an irrational number. We want to show that $R + I$ is irrational.
    - Suppose s represents the sum of R and I ($s = R + I$) and suppose s is rational.
    - If s is rational, then $s + \text{-}R$ would also be rational, because the sum of two rational numbers is rational.
    - $s + \text{-}R$ is not rational, however, because $(R + I) + \text{-}R = I$.
    - $s + \text{-}R$ cannot be both rational and irrational, which means that our original assumption that s was rational was incorrect. s, which is the sum of a rational number and an irrational number, must be irrational.
- The product of a non-zero rational number and an irrational number is irrational. We can show why this is true in a similar way:
    - Let R be rational and I irrational. We want to show that $R \cdot I$ is irrational.
    - Suppose p is the product of R and I ($p = R \cdot I$) and suppose p is rational.
    - If p is rational, then $p \cdot \frac{1}{R}$ would also be rational because the product of two rational numbers is rational.
    - $p \cdot \frac{1}{R}$ is not rational, however, because $R \cdot I \cdot \frac{1}{R} = I$.
    - $p \cdot \frac{1}{R}$ cannot be both rational and irrational, which means our original assumption that p was rational was false. p, which is the product of a rational number and an irrational number, must be irrational.