Lesson 6: Using Diagrams to Find the Number of Groups

Goals

- Explain (orally) how to create a tape diagram to represent and solve a problem asking "How many groups?"
- Justify (orally and using other representations) the answer to a problem asking "How many groups?" in which the divisor is a non-unit fraction and the quotient is a fraction greater than 1.

Learning Targets

- I can use a tape diagram to represent equal-sized groups and find the number of groups.

Lesson Narrative

This is the second lesson in a series of three lessons exploring the "how many groups?" interpretation of division in situations involving fractions.
In the preceding lesson and in this one, the number of groups in each given situation is 1 or greater. In the next lesson, students find the number of groups that is less than 1 ("what fraction of a group?").

Students have used different diagrams to represent multiplication and division. In this lesson, tape diagrams are spotlighted and used more explicitly. They are more abstract and more flexible than other representations students may have chosen for thinking about division problems that involve fractions. Because they use measurement along the length of the tape, tape diagrams are closer to the number line representation of fractions, and ultimately help students visualize division problems on the number line. (Students are not required to do that in this lesson, however.)

Students continue to make the journey from reasoning with concrete quantities to reasoning with abstract representations of fraction
division (MP2).

Alignments

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR2: Collect and Display
- Think Pair Share

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let's draw tape diagrams to think about division with fractions.

6.1 How Many of These in That?

Warm Up: 5 minutes
Students have previously seen tape diagrams used to represent equal-sized groups. This warm-up gives students a chance to create tape diagrams to represent division expressions in a scaffolded way. Each tape is started on a grid and pre-labeled with the known quantity. Each grid square represents 1.

Addressing

- 6.NS.A.1

Instructional Routines

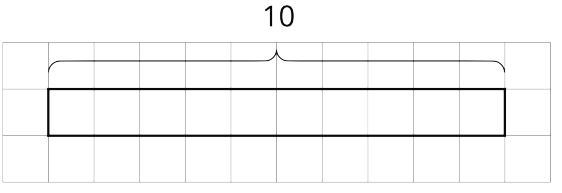
- Think Pair Share

Launch

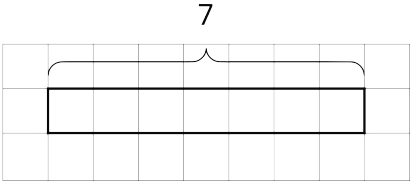
Arrange students in groups of 2. Give students 2 minutes of quiet work time and another minute to share their diagrams with their partner.

Student Task Statement

1. We can think of the division expression $10 \div 2\frac{1}{2}$ as the question: "How many groups of $2\frac{1}{2}$ are in 10?" Complete the tape diagram to represent this question. Then find the answer.

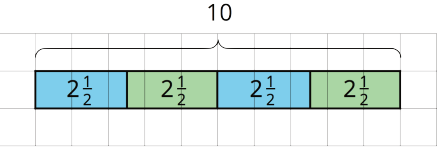


2. Complete the tape diagram to represent the question: "How many groups of 2 are in 7?" Then find the answer.

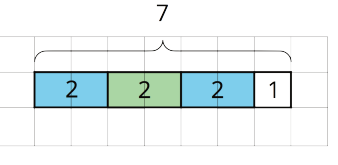


Student Response

1. There are 4 groups of $2\frac{1}{2}$ in 10.



2. There are $3\frac{1}{2}$ groups of 2 in 7.



Activity Synthesis

Select a few students to share their diagrams and answers. After each person shares, poll the class to see if others agree or disagree.

Discuss questions such as:

- "How did you know how large each part of the diagram should be?"
  (In the first question, the length of the tape represents 10, and there are 10 grid squares, so each grid represents 1. Since the size of each group is $2\frac{1}{2}$, each part needs to have $2\frac{1}{2}$ squares.)

- "In the second question, we see three groups of 2 and an extra square of 1. How did you know that the 1 is $\frac{1}{2}$ of a group and not $\frac{1}{7}$ of a group?" (The question asks "how many groups of 2 . . .," so the size of each group is 2, not 7.)

Tell students they will use tape diagrams to help solve other division problems.

6.2 Representing Groups of Fractions with Tape Diagrams

20 minutes

In this lesson, students transition away from using concrete tools (pattern blocks and fraction strips) for reasoning about division to using a tool that is more abstract and more flexible. To represent "how many groups?" or "how many of this in that?" questions, they draw tape diagrams on a grid.

Students continue to write multiplication and division equations to make sense of given situations, but here, they also think about the reasonableness of their answers. They see that the solution to a division problem can be checked using the corresponding multiplication equation.

Some students may write answers as fractions and others as mixed numbers. Both are acceptable. Depending on the situation, one may be more useful than the other. For example, in a measurement situation, a mixed number is easier to interpret, but if we need to perform further calculations with an answer, a fraction is easier to work with. In the course of the unit, students should become comfortable with both forms and be flexible in using them.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR2: Collect and Display

Launch

Arrange students in groups of 3–4. Tell students that they have solved "how many groups?" and "how many of this in that?" problems using pattern blocks and fraction strips. Now they will solve them by drawing tape diagrams.

Give students 2–3 minutes to discuss the first question and write their response. Follow with a brief whole-class discussion. Select 1–2 students to explain how a tape diagram shows us how many $\frac{2}{3}$s are in 1.

Then, give students 8–10 minutes of quiet work time to complete the rest of the task. Ask them to discuss their diagrams only after attempting at least 2 of the 3 remaining questions. Provide access to colored pencils. Some students may find it helpful to identify whole groups and partial groups on a tape diagram by coloring.

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge. For each equation, provide students with a blank template of a tape diagram for students to complete and find the unknown quantities.
Supports accessibility for: Visual-spatial processing; Organization

Access for English Language Learners

Representing, Conversing: MLR2 Collect and Display. As groups discuss the first question, circulate and record language students use to explain how Andre's tape diagram can be used to solve the equation. Listen for phrases such as "equal parts," "same size," and "group of $\frac{2}{3}$s." Consider asking "How are the number of groups represented in the tape diagram?", "How are the values in the equation represented in the diagram?" or "What do the blue and white parts represent?" Display the collected language for all to see and remind students to refer to it throughout the rest of the lesson. This will help students develop mathematical language to explain how a tape diagram can be used to solve a division problem.
Design Principle(s); Maximize meta-awareness

Anticipated Misconceptions

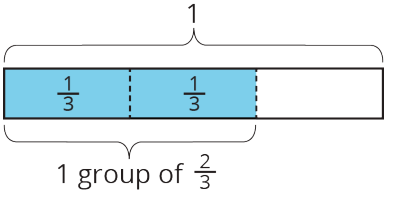
Students may misinterpret the given diagram and think that the answer can only be a whole number because in the diagram, "1 group" is shaded, but the fraction of a group is not. Remind students of their work with pattern blocks in which we saw that the answer to the question "how many of this in that?" or "how many rhombuses are in a trapezoid" could involve a whole number and a fraction.

Student Task Statement

To make sense of the question "How many $\frac{2}{3}$s are in 1?," Andre wrote equations and drew a tape diagram.

$$? \cdot \frac{2}{3} = 1$$

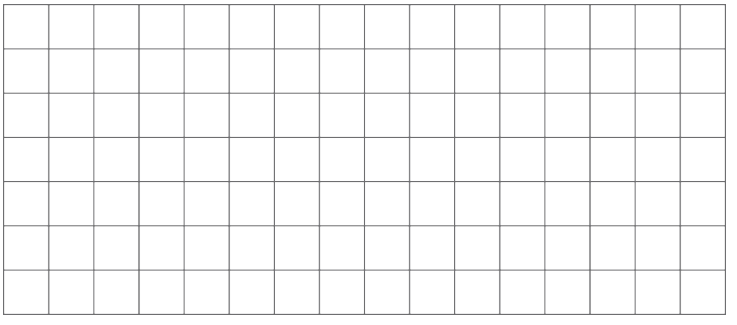
$$1 \div \frac{2}{3} = ?$$



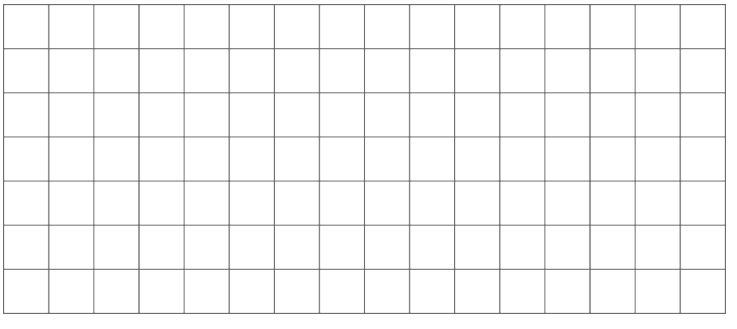
1. In an earlier task, we used pattern blocks to help us solve the equation $1 \div \frac{2}{3} = ?$. Explain how Andre's tape diagram can also help us solve the equation.

2. Write a multiplication equation and a division equation for each question. Then, draw a tape diagram and find the answer.

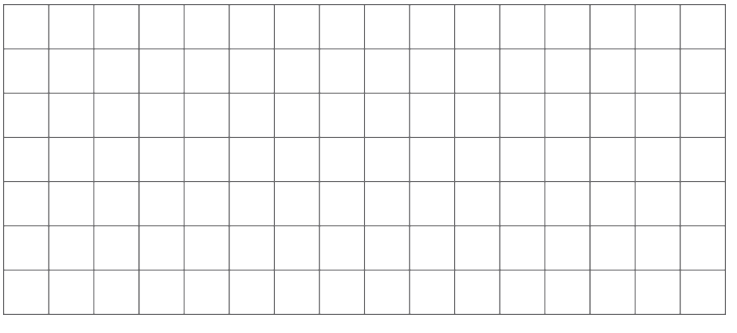
   a. How many $\frac{3}{4}$s are in 1?



   b. How many $\frac{2}{3}$s are in 3?

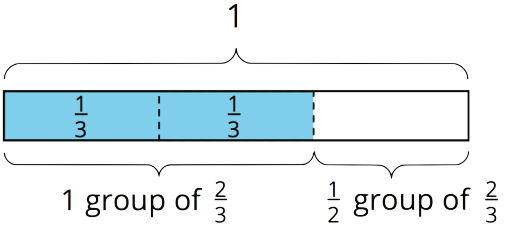


   c. How many $\frac{3}{2}$s are in 5?

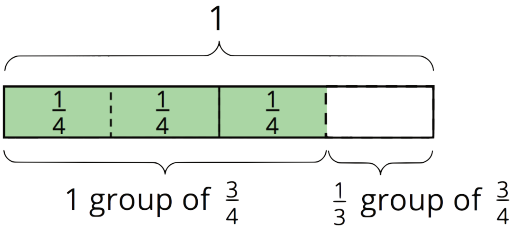


Student Response

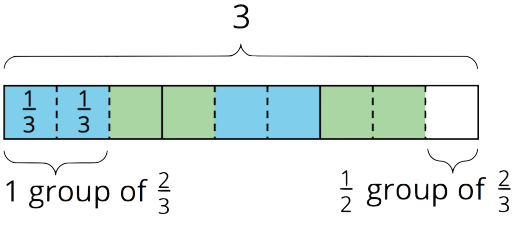
1. Answers vary. Sample reasoning: The tape diagram shows 1 whole broken into 3 equal parts. Each part is $\frac{1}{3}$, so two parts make $\frac{2}{3}$. There is one group of size $\frac{2}{3}$, plus an extra $\frac{1}{3}$, which is half of a group of the same size. This means there are $1\frac{1}{2}$ groups of size $\frac{2}{3}$ in 1, or $1 \div \frac{2}{3} = 1\frac{1}{2}$.



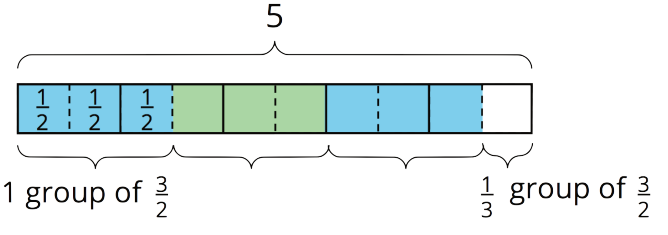
2. a. Multiplication equation: $? \cdot \frac{3}{4} = 1$ (or $\frac{3}{4} \cdot ? = 1$), division equation: $1 \div \frac{3}{4} = ?$ (or $1 \div ? = \frac{3}{4}$), solution: $\frac{4}{3}$ (or $1\frac{1}{3}$).



b. Multiplication equation: $? \cdot \frac{2}{3} = 3$ (or $\frac{2}{3} \cdot ? = 3$), division equation: $3 \div \frac{2}{3} = ?$ (or $3 \div ? = \frac{2}{3}$), solution: $\frac{9}{2}$ (or $4\frac{1}{2}$).



c. Multiplication equation: $? \cdot \frac{3}{2} = 5$ (or $\frac{3}{2} \cdot ? = 5$), division equation: $5 \div \frac{3}{2} = ?$ (or $5 \div ? = \frac{3}{2}$), solution: $\frac{10}{3}$ (or $3\frac{1}{3}$).



Activity Synthesis

Focus the whole-class discussion on two goals: reflecting on the process of creating and using the diagrams, and discussing how we can check our solutions.

For each problem in the second question, select a student to share their response and ask how many others had the same diagram.
If students' diagrams cannot be easily displayed for all to see, consider showing the ones in the Possible Responses. To help students reflect on their process, discuss:

- "How did you begin the diagram? How did you know how to partition the pieces in the diagram?"
- "Which of the two equations—multiplication or division—was helpful in setting up the diagram? How so?"
- "How did you determine how many groups there are?"

- "Can we use pattern blocks to represent these situations? Which might be preferable and why?"

Reinforce the idea that the size of 1 group is what we use as the unit for counting and to find out how many groups there are.

To prompt students to think about the reasonableness of their answers, ask: "How would you know if your answer is correct?" If not mentioned by students, point out how to use multiplication to check their solution to the division problem. For example:

- We wrote $? \cdot \frac{2}{3} = 1$ to represent "how many $\frac{2}{3}$ are in 1?".
- We found the answer to be $1\frac{1}{2}$, so we can substitute $1\frac{1}{2}$ for the "?" and see if $1\frac{1}{2} \cdot \frac{2}{3}$ is indeed 1.
- $1\frac{1}{2} = \frac{3}{2}$, so we can rewrite that expression as $\frac{3}{2} \cdot \frac{2}{3}$.
- $\frac{3}{2} \cdot \frac{2}{3} = 1$, so the answer is correct.

For classes using the digital materials, consider demonstrating an applet to represent a division problem https://ggbm.at/atUteypU. This video shows how to use the tool to answer the question "How many $\frac{1}{2}$ are in 3?" https://vimeo.com/184879045.

6.3 Finding Number of Groups

15 minutes
In this activity, students apply the reasoning and strategies from the previous activity to solve division problems in context. Though the instructions do not prompt students to draw tape diagrams, students may find them to be a handy option.

Students may also choose to express each whole-number or mixed-number dividend as a fraction. For example, they may express the 6 inches as $\frac{48}{8}$ inches and then see how many $\frac{3}{8}$s are in $\frac{48}{8}$. As students work, notice the different strategies they use.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR1: Stronger and Clearer Each Time

Launch

Keep students in the same groups. Give students 8–10 minutes of quiet work time followed by a few minutes to discuss their work with their group.

Some students may wish to use graph paper to draw tape diagrams and colored pencils to mark up parts of the diagrams. Provide access to geometry toolkits. Encourage students to think about the tools and strategies at their disposal and to check their solutions using multiplication.

Student Task Statement

1. Write a multiplication equation or a division equation for each question. Then, find the answer and explain or show your reasoning.

   a. How many $\frac{3}{8}$-inch thick books make a stack that is 6 inches tall?

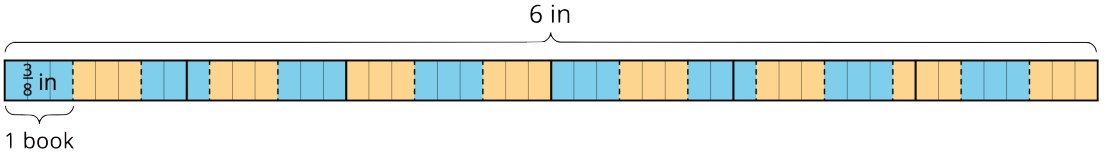
   b. How many groups of $\frac{1}{2}$ pound are in $2\frac{3}{4}$ pounds?

2. Write a question that can be represented by the division equation $5 \div 1\frac{1}{2} = ?$. Then, find the answer and explain or show your reasoning.

Student Response

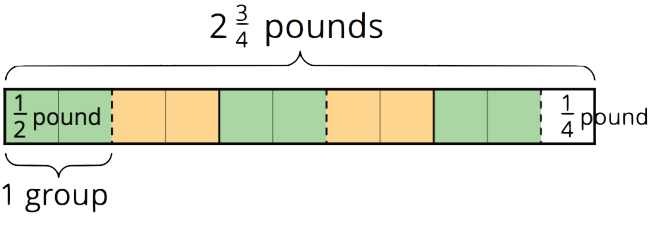
1. a. Multiplication equation: $? \cdot \frac{3}{8} = 6$ (or $\frac{3}{8} \cdot ? = 6$), Division equation: $6 \div \frac{3}{8} = ?$ (or $6 \div ? = \frac{3}{8}$).

   Answer: 16 books. Sample reasoning:



   b. Multiplication equation: $? \cdot \frac{1}{2} = 2\frac{3}{4}$ (or $\frac{1}{2} \cdot ? = 2\frac{3}{4}$), Division equation: $2\frac{3}{4} \div \frac{1}{2} = ?$ (or $2\frac{3}{4} \div ? = \frac{1}{2}$).

   Answer: $5\frac{1}{2}$ groups. Sample reasoning:



2. Questions vary. Sample response: How many $1\frac{1}{2}$ hours are in 5 hours?

   Answer: $3\frac{1}{3}$.

Activity Synthesis

Because the activity is intended to help students reason about division problems quantitatively and abstractly, consider wrapping it up by asking students to write a brief reflection about their reasoning. Display the following prompts for all to see and ask students to write their response on an index card or a sheet of paper so it can be collected.

"A friend is unsure what $2 \div \frac{4}{5}$ means and isn't sure how to find its value. How would you help your friend make sense of the expression? How do you think about it? Share two ways that you find helpful for reasoning about an expression like this."

Access for English Language Learners

Writing, Conversing: MLR1 Stronger and Clearer Each Time. Use this routine to help students improve their writing, by providing them with multiple opportunities to clarify their explanations through conversation. Give students time to meet with 2–3 partners to share and get feedback on their response to the synthesis prompt. Display feedback prompts that will help students strengthen their ideas and clarify their language. For example, "Can you describe that a different way?" and "Can you use both a tape diagram and multiplication equation to explain this?" Give students 1–2 minutes to revise their writing based on the feedback they receive before moving on to the whole-class discussion.
Design Principle(s): Optimize output; Cultivate conversation

Lesson Synthesis

In this lesson, we used tape diagrams to find the number of groups in division situations involving fractions. Consider asking some of the following questions:

- "In the question 'how many $\frac{3}{4}$s are in 6?' (or $6 \div \frac{3}{4} = ?$), what does each number represent?" ($\frac{3}{4}$ represents the size of each group. 6 is the total amount.)
- "What would a tape diagram for this situation show?" (A total tape length representing 6, broken into parts of $\frac{3}{4}$ each.)
- "How does the diagram help us answer the question?" (It allows us to see and count the number of groups.)
- "What if that length cannot be broken equally into $\frac{3}{4}$s? How do we deal with the remainder?" (We can think of it as a partial group. We can compare it to the size of 1 group and see what fraction of a group it is.)
- "We have used pattern blocks, fraction strips, and other diagrams to help us think about division with fractions. How are tape diagrams and these other tools alike?" (They all allow us to represent equal-sized groups so we can see the relationships between the numbers.)

- "How are tape diagrams different than those tools for reasoning about, say, $6 \div \frac{3}{4}$ or $18 \div \frac{2}{5}$?" (Tape diagrams may be more practical since we can specify what the length of the tape and what each part represent. If we use pattern blocks, we may need a lot of blocks to represent certain numbers. If we use fraction strips, we may need to draw much longer strips. Also, not all fractions are shown on the strips, so we may need to add them.)

6.4 How Many in 2?

Cool Down: 5 minutes

This cool-down assesses students' ability to use diagrams to reason about division situations. Two questions are given here, but if time is limited, consider asking students to answer only the second question.

Addressing

- 6.NS.A.1

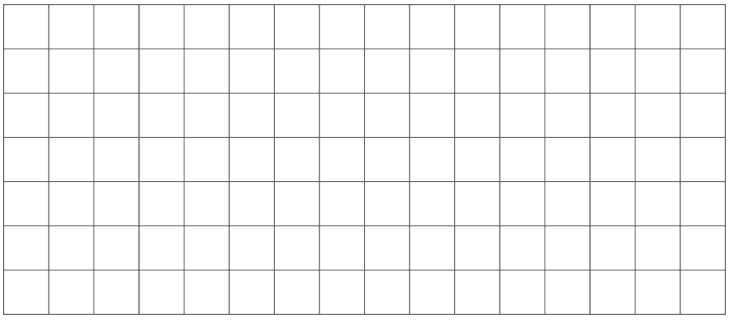
Launch

Provide continued access to colored pencils.

Student Task Statement

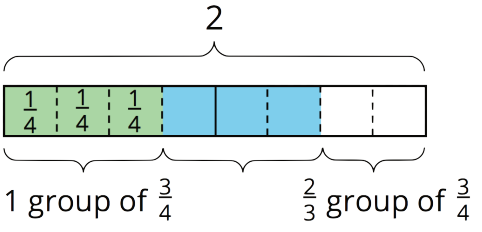
Consider the problem: How many $\frac{3}{4}$s are in 2?

1. Write a multiplication equation and a division equation that can be used to answer the question.
2. Draw a tape diagram and answer the question. Use the grid to help you draw, if needed.



Student Response

1. $? \cdot \frac{3}{4} = 2$. $2 \div \frac{3}{4} = ?$
2. There are two and two-thirds $\frac{3}{4}$s in 2.



Student Lesson Summary

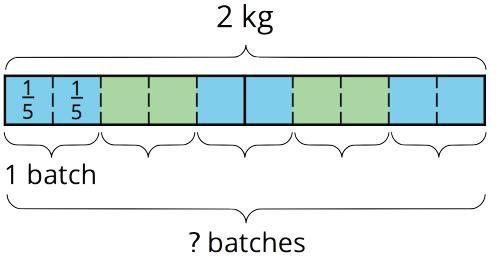
A baker used 2 kilograms of flour to make several batches of a pastry recipe. The recipe called for $\frac{2}{5}$ kilogram of flour per batch. How many batches did she make?

We can think of the question as: "How many groups of $\frac{2}{5}$ kilogram make 2 kilograms?" and represent that question with the equations:

$$? \cdot \frac{2}{5} = 2$$

$$2 \div \frac{2}{5} = ?$$

To help us make sense of the question, we can draw a tape diagram. This diagram shows 2 whole kilograms, with each kilogram partitioned into fifths.



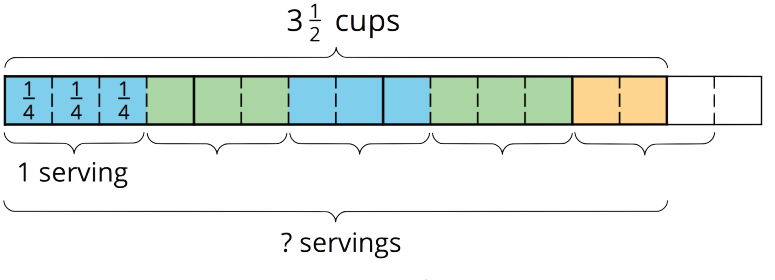
We can see there are 5 groups of $\frac{2}{5}$ in 2. Multiplying 5 and $\frac{2}{5}$ allows us to check this answer: $5 \cdot \frac{2}{5} = \frac{10}{5}$ and $\frac{10}{5} = 2$, so the answer is correct.

Notice the number of groups that result from $2 \div \frac{2}{5}$ is a whole number. Sometimes the number of groups we find from dividing may not be a whole number. Here is an example:

Suppose one serving of rice is $\frac{3}{4}$ cup. How many servings are there in $3\frac{1}{2}$ cups?

$$? \cdot \frac{3}{4} = 3\frac{1}{2}$$

$$3\frac{1}{2} \div \frac{3}{4} = ?$$



Looking at the diagram, we can see there are 4 full groups of $\frac{3}{4}$, plus 2 fourths. If 3 fourths make a whole group, then 2 fourths make $\frac{2}{3}$ of a group. So the number of servings (the "?" in each equation) is $4\frac{2}{3}$. We can check this by multiplying $4\frac{2}{3}$ and $\frac{3}{4}$.

$4\frac{2}{3} \cdot \frac{3}{4} = \frac{14}{3} \cdot \frac{3}{4}$, and $\frac{14}{3} \cdot \frac{3}{4} = \frac{14}{4}$, which is indeed equivalent to $3\frac{1}{2}$.

Lesson 6 Practice Problems

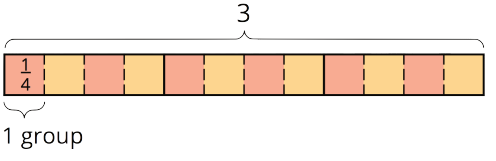
Problem 1

Statement

We can think of $3 \div \frac{1}{4}$ as the question "How many groups of $\frac{1}{4}$ are in 3?" Draw a tape diagram to represent this question. Then find the answer.

Solution

12. Sample diagram:



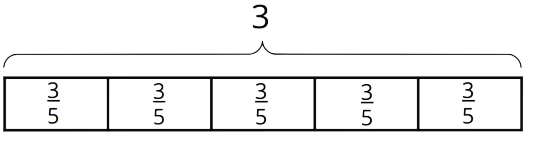
Problem 2

Statement

Describe how to draw a tape diagram to represent and answer $3 \div \frac{3}{5} = ?$ for a friend who was absent.

Solution

Answers vary. Sample explanation: Draw a rectangle whose length represents 3. Partition it into 3 equal parts to show 3 groups of 1. Partition each 1 whole into 5 fifths. There are 15 fifths in 3. Shade each group of 3 fifths, then count how many groups there are in 3.



Problem 3

Statement

How many groups of $\frac{1}{2}$ day are in 1 week?

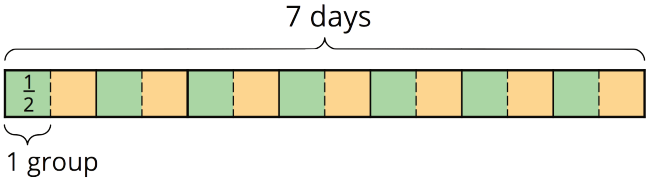
a. Write a multiplication equation or a division equation to represent the question.

b. Draw a tape diagram to show the relationship between the quantities and to answer the question. Use graph paper, if needed.

Solution

a. $? \cdot \frac{1}{2} = 7$ (or equivalent), $7 \div \frac{1}{2} = ?$

b. There are 14 groups of $\frac{1}{2}$-day in a week. Sample diagram:



Problem 4

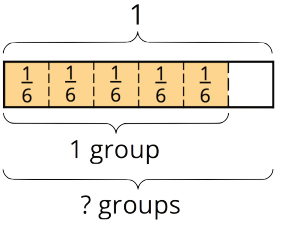
Statement

Diego said that the answer to the question "How many groups of $\frac{5}{6}$ are in 1?" is $\frac{6}{5}$ or $1\frac{1}{5}$. Do you agree with him? Explain or show your reasoning.

Solution

Agree. Sample reasonings:

- $\frac{6}{5} \cdot \frac{5}{6} = \frac{30}{30}$, which equals 1.
- There are 6 sixths in 1. We can make 1 group of $\frac{5}{6}$s and have $\frac{1}{6}$ remaining. $\frac{1}{6}$ is one fifth of $\frac{5}{6}$, so there are $1\frac{1}{5}$ groups of $\frac{5}{6}$ in 1.
-



Problem 5

Statement

Select **all** the equations that can represent the question: "How many groups of $\frac{4}{5}$ are in 1?"

A. $? \cdot 1 = \frac{4}{5}$

B. $1 \cdot \frac{4}{5} = ?$

C. $\frac{4}{5} \div 1 = ?$

D. $? \cdot \frac{4}{5} = 1$

E. $1 \div \frac{4}{5} = ?$

Solution

["D", "E"]
(From Unit 4, Lesson 5.)

Problem 6

Statement

Calculate each percentage mentally.

a. What is 10% of 70?

b. What is 10% of 110?

c. What is 25% of 160?

d. What is 25% of 48?

e. What is 50% of 90?

f. What is 50% of 350?

g. What is 75% of 300?

h. What is 75% of 48?

Solution

a. 7

b. 11

c. 40

d. 12

e. 45

f. 175

g. 225

h. 36

(From Unit 3, Lesson 14.)