Lesson 10: Dividing by Unit and Non-Unit Fractions

Goals

- Interpret and critique explanations (in spoken and written language, as well as in other representations) of how to divide by a fraction.
- Use a tape diagram to represent dividing by a non-unit fraction $\frac{a}{b}$ and explain (orally) why this produces the same result as multiplying the number by b and dividing by a.
- Use a tape diagram to represent dividing by a unit fraction $\frac{1}{b}$ and explain (orally and in writing) why this is the same as multiplying by b.

Learning Targets

- I can divide a number by a non-unit fraction $\frac{a}{b}$ by reasoning with the numerator and denominator, which are whole numbers.
- I can divide a number by a unit fraction $\frac{1}{b}$ by reasoning with the denominator, which is a whole number.

Lesson Narrative

This is the first of two lessons in which students pull together the threads of reasoning from the previous six lessons to develop a general algorithm for dividing fractions. Students start by recalling the idea from grade 5 that dividing by a unit fraction has the same outcome as multiplying by the reciprocal of that unit fraction. They use tape diagrams to verify this.

Next, they use the same diagrams to look at the effects of dividing by non-unit fractions. Through repetition, they notice a pattern in the steps of their reasoning (MP8) and structure in the visual representation of these steps (MP7). Students see that division by a non-unit fraction can be thought of as having two steps: dividing by the unit fraction, and then dividing the result by the numerator of the fraction. In other words, to divide by $\frac{2}{5}$ is equivalent to dividing by $\frac{1}{5}$, and then again by 2. Because dividing by a unit fraction $\frac{1}{5}$ is equivalent to multiplying by 5, we can evaluate division by $\frac{2}{5}$ by multiplying by 5 and dividing by 2.

Alignments

Building On

- 5.NF.B: Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- MLR3: Clarify, Critique, Correct
- MLR8: Discussion Supports
- Think Pair Share

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let's look for patterns when we divide by a fraction.

10.1 Dividing by a Whole Number

Warm Up: 10 minutes
In this warm-up, students use tape diagrams to revisit the idea that dividing by a whole number is equivalent to multiplying by a unit fraction. Though this is a review of a grade 5 expectation, connecting the division problems to diagrams allows students to see the equivalence in the related division and multiplication problems. It also prepares students to extend the reasoning and representations used here to division by non-unit fractions later.

Building On

- 5.NF.B

Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. Ask one person in each group to draw diagrams and answer the questions for Partner A, and the other to take on the questions for Partner B. Give students a few minutes of quiet time to complete the first two questions, and then ask them to collaborate on the last two.

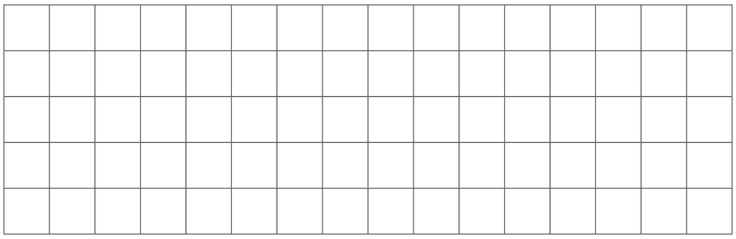
Student Task Statement

Work with a partner. One person solves the problems labeled “Partner A” and the other person solves those labeled “Partner B.” Write an equation for each question. If you get stuck, consider drawing a diagram.

1. Partner A:

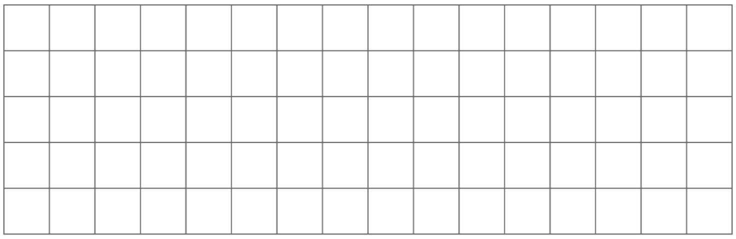
   How many 3s are in 12?

   Division equation:



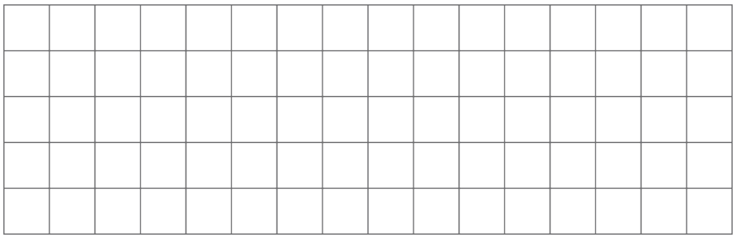
   How many 4s are in 12?

   Division equation:



   How many 6s are in 12?

   Division equation:



   Partner B:

What is 12 groups of $\frac{1}{3}$?

Multiplication equation:

What is 12 groups of $\frac{1}{4}$?

Multiplication equation:

What is 12 groups of $\frac{1}{6}$?

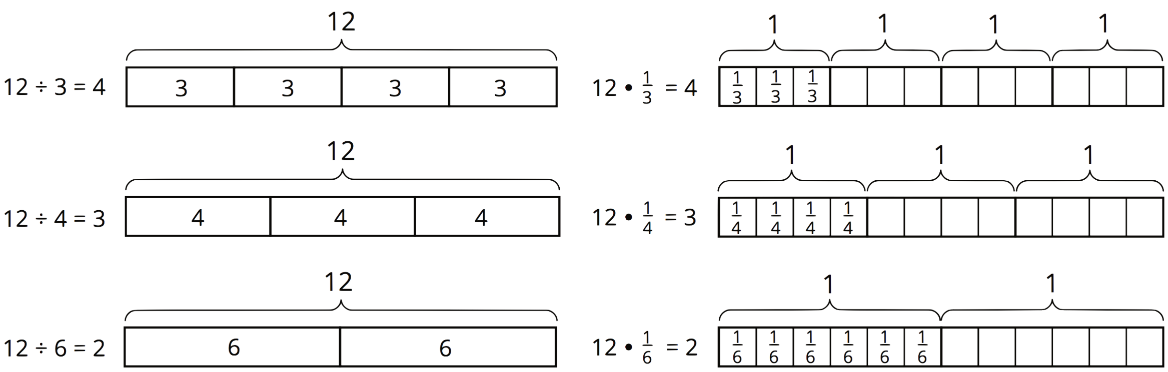
Multiplication equation:

2. What do you notice about the diagrams and equations? Discuss with your partner.
3. Complete this sentence based on what you noticed:

   Dividing by a whole number a produces the same result as multiplying by ________.

Student Response

1.

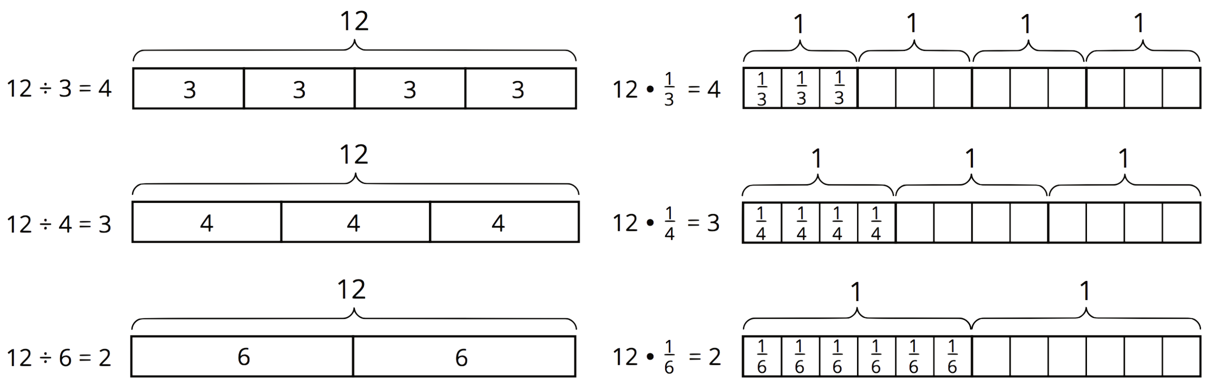


2. Answers vary. Sample responses:
    - The quotients in the division equations have the same value as the products in the multiplication equations.
    - Both sets of problems use the number 12. The division problems have divisors 3, 4, and 6, and the multiplication problems have a factor that is the reciprocal of those numbers ($\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$).
    - Each pair of diagrams are divided into the same number of major parts but they show different information.
    - Dividing by a whole number gives the same result as multiplying by its reciprocal.
3. Dividing by a whole number a produces the same result as multiplying by $\frac{1}{a}$.

Activity Synthesis

Invite a couple of students to share their observations about their group's diagrams and answers. Students should notice that the answers for the three division problems match those for the multiplication ones, even though the questions were not the same and their diagrams show groups of different sizes. Ask a few students to share their response to the last question.

Consider displaying the following image to reinforce the idea that dividing by a whole number has the same effect as multiplying by the reciprocal of that number.



10.2 Dividing by Unit Fractions

15 minutes

In this activity, students use tape diagrams and the meanings of division to divide a number by unit fractions. They do this as a first step toward generalizing the reasoning for dividing any two fractions. By reasoning repeatedly and noticing a pattern (MP8), students arrive at the conclusion that $a \div \frac{1}{b}$ is equivalent to $a \cdot b$.

Later, students put together those results to see that dividing by a fraction $\frac{c}{d}$ is equivalent to dividing by c and multiplying by d.

As students work, notice those who are able to transfer the reasoning in the first two problems to subsequent problems without the help of diagrams. Select several students to share later.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR8: Discussion Supports
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 5–6 minutes of quiet work time and then time to share their work with a partner. Provide access to colored pencils. Some students may find it helpful to identify whole groups and partial groups on a tape diagram by coloring.

For classes using the digital materials, an applet from an earlier lesson can also be used to represent the division problems here,

Video 'Fraction Division Tool Demo' available here: https://player.vimeo.com/video/304136674.

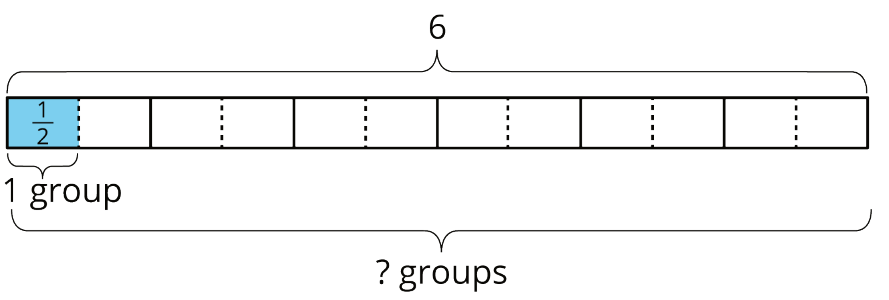
Anticipated Misconceptions

For some students, the structure in the division may still not be apparent by the time they get to problems such as $6 \div \frac{1}{25}$, and they may try to partition a section in a tape diagram into 25 parts. Ask them to study the diagrams as well as the answers to the previous questions carefully and to look for a pattern.

Student Task Statement

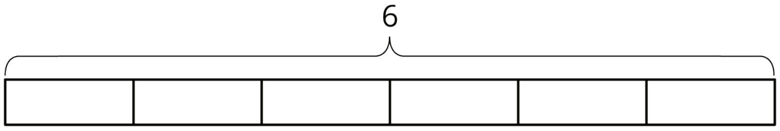
To find the value of $6 \div \frac{1}{2}$, Elena thought, "How many $\frac{1}{2}$s are in 6?" and then she drew this tape diagram. It shows 6 ones, with each one partitioned into 2 equal pieces.

$6 \div \frac{1}{2}$



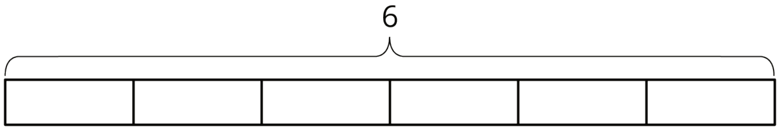
1. For each division expression, complete the diagram using the same method as Elena. Then, find the value of the expression.

a. $6 \div \frac{1}{3}$



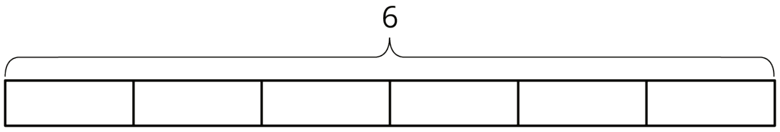
Value of the expression: ___________

b. $6 \div \frac{1}{4}$



Value of the expression: ___________

c. $6 \div \frac{1}{6}$



Value of the expression: ___________

2. Examine the expressions and answers more closely. Look for a pattern. How could you find how many halves, thirds, fourths, or sixths were in 6 without counting all of them? Explain your reasoning.

3. Use the pattern you noticed to find the values of these expressions. If you get stuck, consider drawing a diagram.

   a. $6 \div \frac{1}{8}$

   b. $6 \div \frac{1}{10}$

   c. $6 \div \frac{1}{25}$

   d. $6 \div \frac{1}{b}$

4. Find the value of each expression.

   a. $8 \div \frac{1}{4}$

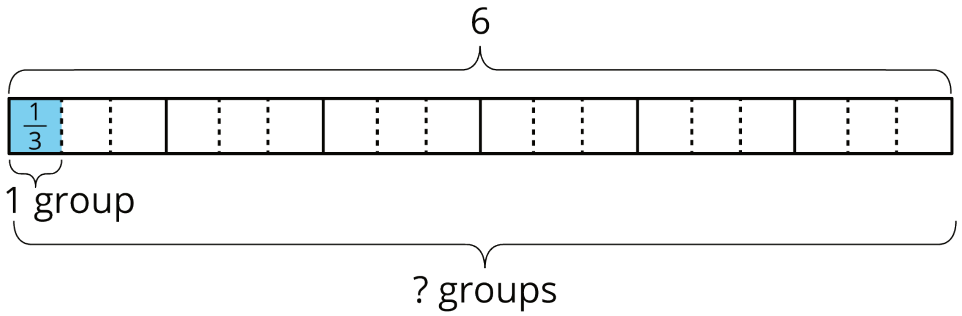
   b. $12 \div \frac{1}{5}$

   c. $a \div \frac{1}{2}$

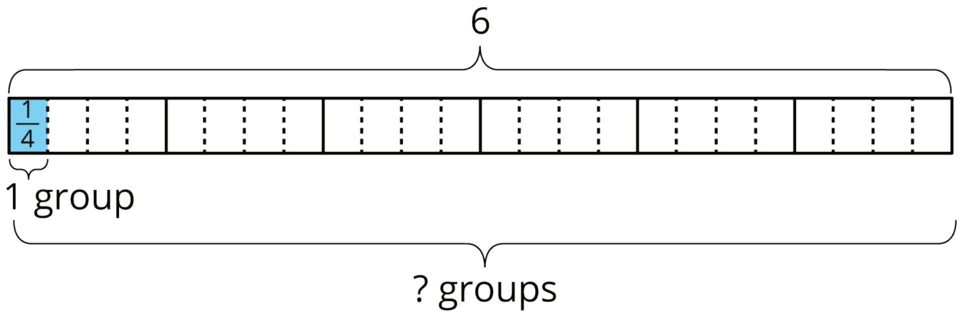
   d. $a \div \frac{1}{b}$

Student Response

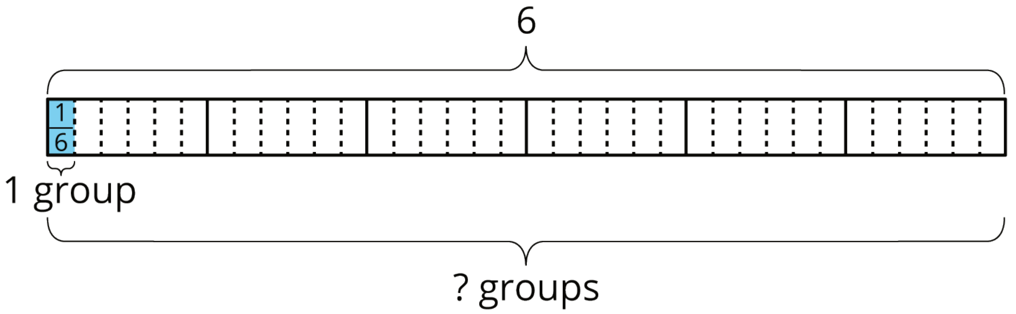
1. a. Value: 18



b. Value:24



c. Value: 36



2. Answers vary. Sample response: I divided each 1 whole in the tape diagram into the same number of pieces as in the number in the denominator. If the fraction was $\frac{1}{4}$, I broke each one into 4 pieces. To find how many pieces are in 6, I multiplied 6 by the denominator. If the fraction is $\frac{1}{4}$, there are 4 times as many pieces on my tape diagram as in the original tape diagram.

3. a. 48

   b. 60

   c. 150

   d. $6 \cdot b$

4. a. 32

b. 60

c. $a \cdot 2$

d. $a \cdot b$

Activity Synthesis

Select previously identified students to share their responses to the questions $6 \div \frac{1}{b}$ and $a \div \frac{1}{b}$. To highlight the connections between the diagram, division by a unit fraction, and multiplication by the reciprocal of the fraction ask students:

- “How is the division by a unit fraction depicted in the diagrams?” (The tape diagram is broken into equal parts. The unit fraction is the size of each part.)
- “Where in the diagrams do we see the multiplication?” (The multiplication shows the number of parts in each 1 whole.)
- “How are the two—the division by a unit fraction and the multiplication—related?” (When we divide a number by a unit fraction $\frac{1}{b}$, we end up with b times as many parts, so dividing by $\frac{1}{b}$ is the same as multiplying by b.)

If not already articulated by students, clarify that dividing a number by a unit fraction has the same result as multiplying by its reciprocal.

Discuss the usefulness and limits of diagrams. Ask:

- “How do we find the value of $1{,}000 \div \frac{1}{9}$ or $2 \div \frac{1}{250}$ using a diagram?” (By dividing each 1 whole into 9 parts or 250 parts)
- “Would you use diagrams to find these quotients? Why or why not?” (No, it is not practical.)
- “When working on the task, did you stop partitioning the tape diagrams at some point? If so, why?”
- “Why do we use diagrams? When can they be helpful?” (Diagrams can show us the structure or relationships between numbers and help us see the general process.)

If not mentioned by students, point out that for larger numbers, or smaller fractions, drawing a full diagram becomes increasingly cumbersome. Noticing the structure that is visible in the diagrams for easier cases allows us to use it for more difficult ones.

Access for English Language Learners

Conversing, Representing: MLR8 Discussion Supports. Use this routine to amplify mathematical language. After students share a response, invite them to repeat their reasoning using mathematical language relevant to the lesson (denominator, repeated, unit fraction, partitioned, reciprocal). For example, say, "Can you say that again, using the phrase 'reciprocal of the fraction'?" Consider inviting remaining students to chorally repeat these phrases to provide additional opportunities for all students to produce language verbally.
Design Principle(s): Support sense-making

10.3 Dividing by Non-unit Fractions

15 minutes
Here students continue to use tape diagrams to reason about division and extend their observations about unit fractions to non-unit fractions. Specifically, they explore how to represent the numerator of the fraction in the tape diagram and study its effect on the quotient. Students generalize their observations as operational steps and then as expressions, which they then use to solve other division problems.

As students work, notice those who effectively show the divisor on their diagrams, i.e., the multiplication by the denominator and division by the numerator, as well as those who could explain why the steps make sense.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR3: Clarify, Critique, Correct

Launch

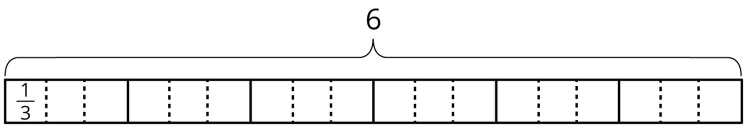
Keep students in groups of 2. Give students 5–7 minutes of quiet work time and then time to share their responses with their partner. Provide continued access to colored pencils.

Anticipated Misconceptions

Some students may read the phrase "partition 1 section into 4 parts" in Elena's reasoning and focus on the value of each small part $\left(\frac{1}{4}\right)$ instead on how many parts are now shown. Similarly, they may take "making of 3 of these parts into one piece" to imply multiplying the $\frac{1}{4}$ by 3, instead of looking at how it changes the number of pieces. Explain that Elena's diagram& suggests that she interpreted $6 \div \frac{3}{4}$ as "how many $\frac{3}{4}$s are in 6?" which tells us that we are looking for the number of groups, rather than the value of each part in the diagram.

Student Task Statement

1. To find the value of $6 \div \frac{2}{3}$, Elena started by drawing a diagram the same way she did for $6 \div \frac{1}{3}$.

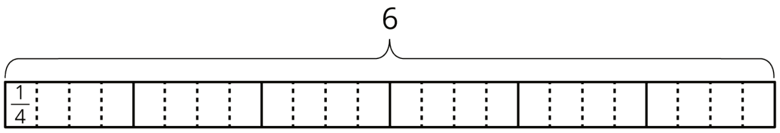


   a. Complete the diagram to show how many $\frac{2}{3}$s are in 6.

   b. Elena says, "To find $6 \div \frac{2}{3}$, I can just take the value of $6 \div \frac{1}{3}$ and then either multiply it by $\frac{1}{2}$ or divide it by 2." Do you agree with her? Explain your reasoning.

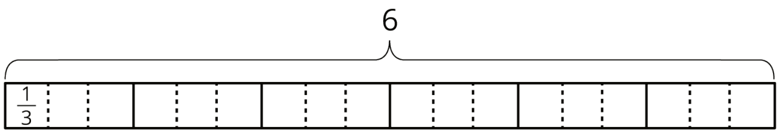
2. For each division expression, complete the diagram using the same method as Elena. Then, find the value of the expression. Think about how you could find that value without counting all the pieces in your diagram.

   a. $6 \div \frac{3}{4}$



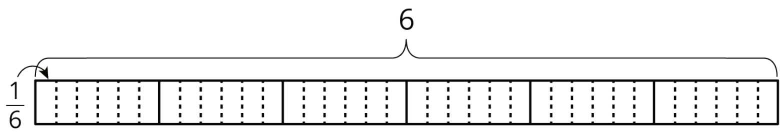
   Value of the expression:__________

   b. $6 \div \frac{4}{3}$



   Value of the expression:__________

   c. $6 \div \frac{4}{6}$



   Value of the expression:__________

3. Elena examined her diagrams and noticed that she always took the same two steps to show division by a fraction on a tape diagram. She said:

   "My first step was to divide each 1 whole into as many parts as the number in the denominator. So if the expression is $6 \div \frac{3}{4}$, I would break each 1 whole into 4 parts. Now I have 4 times as many parts.

My second step was to put a certain number of those parts into one group, and that number is the numerator of the divisor. So if the fraction is $\frac{3}{4}$, I would put 3 of the $\frac{1}{4}$s into one group. Then I could tell how many $\frac{3}{4}$s are in 6."

Which expression represents how many $\frac{3}{4}$s Elena would have after these two steps? Be prepared to explain your reasoning.

- $6 \div 4 \cdot 3$
- $6 \cdot 4 \div 3$
- $6 \div 4 \div 3$
- $6 \cdot 4 \cdot 3$

4. Use the pattern Elena noticed to find the values of these expressions. If you get stuck, consider drawing a diagram.

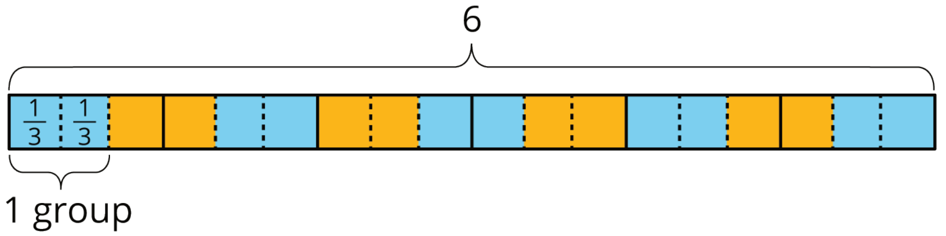
    a. $6 \div \frac{2}{7}$

    b. $6 \div \frac{3}{10}$

    c. $6 \div \frac{6}{25}$

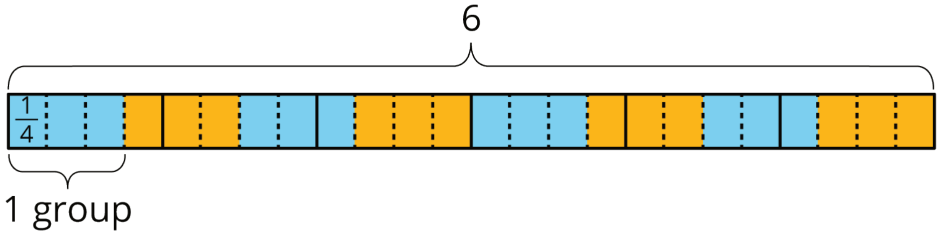
Student Response

1. a.

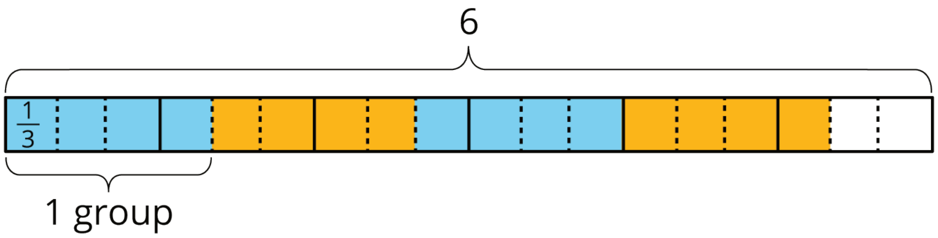


b. I agree. Sample reasoning: $\frac{2}{3}$ is two $\frac{1}{3}$s. If we put two $\frac{1}{3}$s in a group, we would have half as many pieces as we did for $6 \div \frac{1}{3}$.

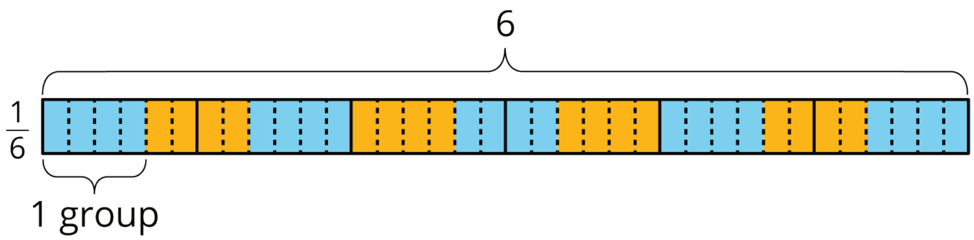
2. a. Value: 8



b. Value: $\frac{9}{2}$ (or $4\frac{1}{2}$)



c. Value: 9



3. $6 \cdot 4 \div 3$. Sample reasoning: Dividing each 1 whole into 4 parts makes 4 times as many pieces in the diagram. Making every 3 of those pieces into a group makes $\frac{1}{3}$ as many groups as there were pieces in the diagram. To get the number of groups, multiply the number of wholes—which is 6—by $\frac{1}{3}$ or divide it by 3.

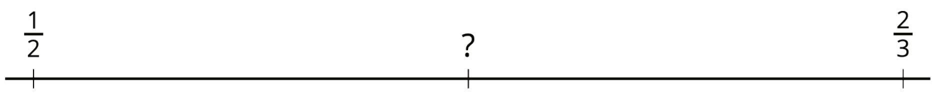
4. a. 21, because $6 \cdot 7 \div 2 = 21$

   b. 20, because $6 \cdot 10 \div 3 = 20$

   c. 25, because $6 \cdot 25 \div 6 = 25$

Are You Ready for More?

Find the missing value.

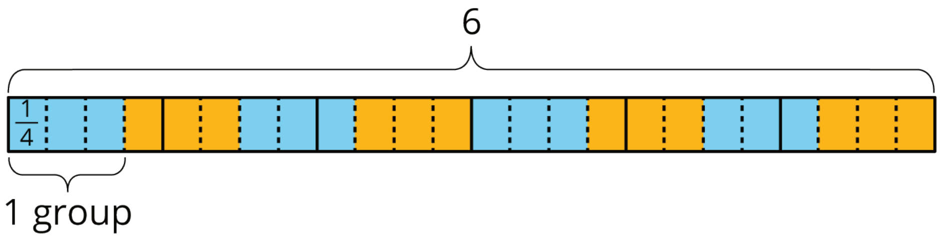


Student Response

$\frac{7}{12}$

Activity Synthesis

Select a few students to share their diagrams and explanations about why Elena's reasoning and method work. Display the following tape diagram for $6 \div \frac{3}{4}$, if needed. Ask students to point out where in the diagram the two steps are visible.



To involve more students in the conversation, consider asking questions such as:

- “Who can restate ___'s reasoning in different words?”
- “Did anyone think about the division the same way but would explain it differently?”
- “Does anyone want to add an observation to the way ____ reasoned about the division?”
- “Do you agree or disagree? Why?”

Highlight that dividing a number c by a fraction $\frac{a}{b}$ has the same result as multiplying by b, then dividing by a (or multiplying by $\frac{1}{a}$).

Access for English Language Learners

Writing, Conversing: MLR3 Clarify, Critique, Correct. Display the following statement: “The value of $6 \div \frac{2}{7}$ is 84 because $6 \div \frac{1}{7}$ is 42. Then I multiplied that by 2 to get 84.” Keep students in groups of 2. Invite groups to consider the reasoning behind the statement, identify any mathematical or language errors they can see, and then write a correct explanation. As groups discuss, listen for the language students use to describe the reasoning they think contributed to this error. Call students' attention to the words and phrases they use that help them clarify their revised statements. This helps students evaluate, and improve on, the written mathematical arguments of others.
Design Principle(s): Support sense-making, Cultivate conversation

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Lesson Synthesis

In this lesson, we reasoned about fraction division using tape diagrams. We traced the steps in our reasoning and analyzed the outcomes.

- “What did we notice about the result of dividing a number by a unit fraction? Can you explain with an example?” (It has the same outcome as multiplying by the denominator of the fraction.)
- “What observations did we make when dividing a number by a non-unit fraction? Can you explain with an example?” (It has the same outcome as multiplying by the denominator of the fraction and dividing by the numerator, and dividing by the numerator is the same as multiplying by its reciprocal.)

- "Suppose we are finding $5 \div \frac{7}{25}$. How might these observations help us find this quotient?" (We can multiply 5 by 25 and then divide by 7, instead of drawing a diagram and breaking each 1 into 25 parts, and so on.)

Emphasize that recognizing the patterns in how we reason about division of simpler fractions can help us divide other fractions more efficiently. Instead of drawing and reasoning with diagrams, which could be time consuming, we can follow the same series of steps to find quotients.

10.4 Dividing by $\frac{1}{3}$ and $\frac{3}{5}$

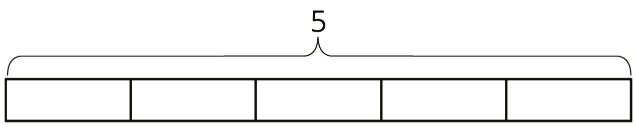
Cool Down: 5 minutes

Addressing

- 6.NS.A.1

Student Task Statement

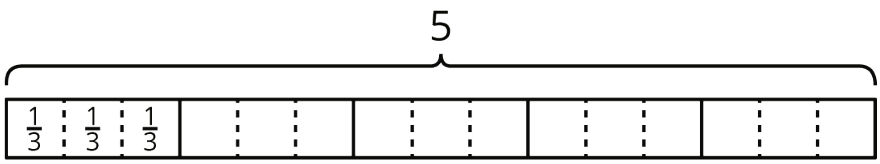
1. Explain or show how you could find $5 \div \frac{1}{3}$. You can use this diagram if it is helpful.



2. Find $12 \div \frac{3}{5}$. Try not to use a diagram, if possible. Show your reasoning.

Student Response

1. Answers vary. Sample reasoning: $5 \div \frac{1}{3}$ can mean "How many $\frac{1}{3}$s (thirds) are in 5?" There are 3 thirds in 1, so in 5, there are 5 times as many thirds. Five times as many is $5 \cdot 3$, so there are 15 thirds in 5.

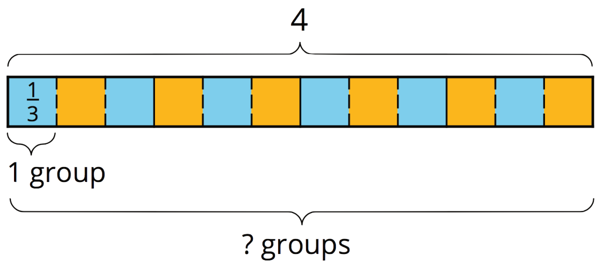


2. 20. Sample reasoning: $12 \div \frac{3}{5} = 12 \cdot 5 \cdot \frac{1}{3} = 20$

Student Lesson Summary

To answer the question "How many $\frac{1}{3}$s are in 4?" or "What is $4 \div \frac{1}{3}$?", we can reason that there are 3 thirds in 1, so there are $(4 \cdot 3)$ thirds in 4.

In other words, dividing 4 by $\frac{1}{3}$ has the same result as multiplying 4 by 3.

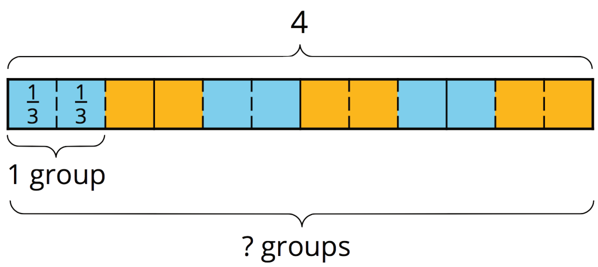


$$4 \div \frac{1}{3} = 4 \cdot 3$$

In general, dividing a number by a unit fraction $\frac{1}{b}$ is the same as multiplying the number by b, which is the **reciprocal** of $\frac{1}{b}$.

How can we reason about $4 \div \frac{2}{3}$?

We already know that there are $(4 \cdot 3)$ or 12 groups of $\frac{1}{3}$s in 4. To find how many $\frac{2}{3}$s are in 4, we need to put together every 2 of the $\frac{1}{3}$s into a group. Doing this results in half as many groups, which is 6 groups. In other words:



$$4 \div \frac{2}{3} = (4 \cdot 3) \div 2$$

or

$$4 \div \frac{2}{3} = (4 \cdot 3) \cdot \frac{1}{2}$$

In general, dividing a number by $\frac{a}{b}$, is the same as multiplying the number by b and then dividing by a, or multiplying the number by b and then by $\frac{1}{a}$.

Glossary

- reciprocal

Lesson 10 Practice Problems

Problem 1

Statement

Priya is sharing 24 apples equally with some friends. She uses division to determine how many people can have a share if each person gets a particular number of apples. For example, $24 \div 4 = 6$ means that if each person gets 4 apples, then 6 people can have apples. Here are some other calculations:

$24 \div 4 = 6$ $\quad$ $24 \div 2 = 12$ $\quad$ $24 \div 1 = 24$ $\quad$ $24 \div \frac{1}{2} = ?$

a. Priya thinks the "?" represents a number less than 24. Do you agree? Explain or show your reasoning.

b. In the case of $24 \div \frac{1}{2} = ?$, how many people can have apples?

Solution

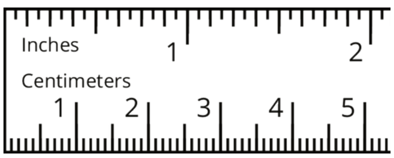
a. Disagree. Sample reasoning:
   - As the amount for each person gets smaller, more people can have apples.
   - There is a pattern in the numbers: when the number of apples per person is halved, the number of people doubles. Since 1 apple per person means 24 people can enjoy an apple, then $\frac{1}{2}$ apple per person means 48 (twice as many) people can enjoy some apple.

b. 48 apples

Problem 2

Statement

Here is a centimeter ruler.



a. Use the ruler to find $1 \div \frac{1}{10}$ and $4 \div \frac{1}{10}$.

b. What calculation did you do each time?

c. Use this pattern to find $18 \div \frac{1}{10}$.

d. Explain how you could find $4 \div \frac{2}{10}$ and $4 \div \frac{8}{10}$.

Solution

a. 10 and 40

b. Each time the dividend was multiplied by 10.

c. 180

d. Take the answer from $4 \div \frac{1}{10}$ and divide it by 2 or 8, getting 20 and 5, respectively.

Problem 3

Statement

Find each quotient.

a. $5 \div \frac{1}{10}$

b. $5 \div \frac{3}{10}$

c. $5 \div \frac{9}{10}$

Solution

a. 50

b. $\frac{50}{3}$ or $16\frac{2}{3}$

c. $\frac{50}{9}$ or $5\frac{5}{9}$

Problem 4

Statement

Use the fact that $2\frac{1}{2} \div \frac{1}{8} = 20$ to find $2\frac{1}{2} \div \frac{5}{8}$. Explain or show your reasoning.

Solution

Explanations vary. Sample response: There are 20 groups of $\frac{1}{8}$ in $2\frac{1}{2}$. If the size of each group is quintupled (from $\frac{1}{8}$ to $\frac{5}{8}$), then the number of groups will decrease by a factor of 5.

Problem 5

Statement

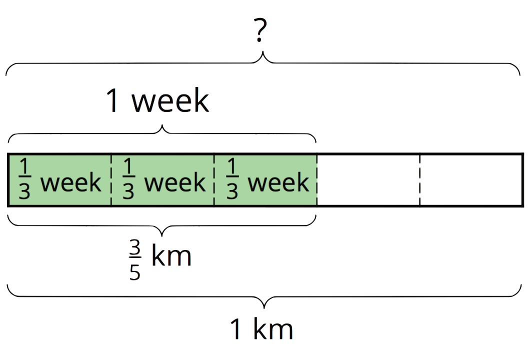
Consider the problem: It takes one week for a crew of workers to pave $\frac{3}{5}$ kilometer of a road. At that rate, how long will it take to pave 1 kilometer?

Write a multiplication equation and a division equation to represent the question. Then find the answer and show your reasoning.

Solution

a. $\frac{3}{5} \cdot ? = 1$ (or equivalent), $1 \div \frac{3}{5} = ?$

b. $1\frac{2}{3}$ weeks. Sample reasoning:



(From Unit 4, Lesson 9.)

Problem 6

Statement

A box contains $1\frac{3}{4}$ pounds of pancake mix. Jada used $\frac{7}{8}$ pound for a recipe. What fraction of the pancake mix in the box did she use? Explain or show your reasoning. Draw a diagram, if needed.

Solution

$\frac{1}{2}$. Sample explanations:

- $1\frac{3}{4}$ is $\frac{7}{4}$. $\frac{7}{8}$ is half of $\frac{7}{4}$.
- The question can be represented with: $? \cdot \frac{7}{4} = \frac{7}{8}$. The "?" has to be $\frac{1}{2}$ so that the product is $\frac{7}{8}$.

(From Unit 4, Lesson 7.)

Problem 7

Statement

Calculate each percentage mentally.

a. 25% of 400

b. 50% of 90

a. 75% of 200

b. 10% of 8,000

a. 5% of 20

Solution

a. 100

b. 45

c. 150

d. 800

e. 1

(From Unit 3, Lesson 14.)