Lesson 7: What Fraction of a Group?

Goals

- Comprehend the phrase “What fraction of a group?” (in spoken and written language) as a variation of the question “How many groups?” that is used when the quotient is less than 1.
- Create a tape diagram to represent and solve a problem asking “How many groups?” in which the quotient is a fraction less than 1.
- Write multiplication and division equations to represent a problem asking “How many times as long?”

Learning Targets

- I can tell when a question is asking for the number of groups and that number is less than 1.
- I can use diagrams and multiplication and division equations to represent and answer “what fraction of a group?” questions.

Lesson Narrative

In the previous three lessons, students explored the “how many groups?” interpretation of division. Their explorations included situations where the number of groups was a whole number or a mixed number. In this lesson, they extend the work to include cases where the number of groups is a fraction less than 1, that is, situations in which the total amount is smaller than the size of 1 group. In such situations, the question becomes “what fraction of a group?”.

Students notice that they can use the same reasoning strategies as in situations with a whole number of groups, because the structure

$$\text{(number of groups)} \boldsymbol{\cdot} \text{(size of a group)} = \text{(total amount)}$$

is the same as before (MP7). They write multiplication equations of this form and for the corresponding division equations.

Throughout the lesson, students practice attending to details (in diagrams, descriptions, or equations) about how the given quantities relate to the size of 1 group.

Alignments

Building On

- 5.NF.B.4: Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- MLR6: Three Reads
- MLR8: Discussion Supports
- Think Pair Share

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

> ### Student Learning Goals
> Let’s think about dividing things into groups when we can’t even make one whole group.

7.1 Estimating a Fraction of a Number

Warm Up: 5 minutes
In this warm-up, students estimate the value of a fraction of a number (e.g., $\frac{1}{3}$ of 7) using what they know about the size of the given fraction. Then, they write multiplication expressions to represent the verbal questions. The goal is to activate prior understandings that a fraction of a number can be found by multiplication, preparing students to explore division problems in which the quotient is less than 1 whole.

Building On

- 5.NF.B.4

Launch

Ask students to keep their materials closed. Display one estimation question at a time. (If all questions are displayed, ask students to work on one question at a time and to begin when cued.) Give students 30 seconds of quiet think time per question and ask them to give a signal when they have an answer and can explain their strategy.

Select 1–2 students to briefly share their estimates and how they made them. Record and display their estimates for all to see. After discussing the third estimation question, ask students to write a multiplication expression to represent each of the three questions.

Anticipated Misconceptions

Some students may try to find the exact answers to the questions instead of estimating. Encourage them to think about benchmark fractions that could help them estimate.

Student Task Statement

1. Estimate the quantities:
    a. What is $\frac{1}{3}$ of 7?
    b. What is $\frac{4}{5}$ of $9\frac{2}{3}$?
    c. What is $2\frac{4}{7}$ of $10\frac{1}{9}$?
2. Write a multiplication expression for each of the previous questions.

Student Response

1. Answers vary. Possible responses:
    a. Less than $3\frac{1}{2}$, because that would be $\frac{1}{2}$ of 7, but greater than 2, because that would be $\frac{1}{3}$ of 6.
    b. A little less than 8, because $9\frac{2}{3}$ is just under 10, and $\frac{4}{5}$ of 10 is 8.
    c. A little more than 25, because $2\frac{4}{7}$ is a bit more than $2\frac{1}{2}$, and $10\frac{1}{9}$ is just a little over 10. $2\frac{1}{2}$ of 10 is 25.
2.
    a. $\frac{1}{3} \cdot 7$ (or $7 \cdot \frac{1}{3}$)
    b. $\frac{4}{5} \cdot 9\frac{2}{3}$ (or $9\frac{2}{3} \cdot \frac{4}{5}$)
    c. $2\frac{4}{7} \cdot 10\frac{1}{9}$ (or $10\frac{1}{9} \cdot 2\frac{4}{7}$)

Activity Synthesis

Ask a few students to share the expressions they wrote for the questions. Record and display the expressions for all to see. Ask the class to indicate if they agree or disagree with each expression.

If not already brought up in students' explanations, highlight the idea that we can find the exact value of a fraction of a number (e.g., $\frac{4}{5}$ of $9\frac{2}{3}$) by multiplying the fraction and the number. It does not matter whether the number is a whole number, a mixed number, or another fraction.

To involve more students in the conversation, consider asking:

- "Who can restate ___'s reasoning in a different way?"

- “Did anyone solve the problem the same way but would explain it differently?”
- “Did anyone solve the problem in a different way?”
- “Does anyone want to add on to _____’s strategy?”
- “Do you agree or disagree? Why?”

7.2 Fractions of Ropes

10 minutes (there is a digital version of this activity)
This task helps to transition students from thinking about “how many groups?” to “what fraction of a group?”.

Students compare different lengths of ropes and express their relative lengths in multiplicative terms. Rope B and C are 5 and $2\frac{1}{2}$ times as long as rope A, respectively, but rope D is shorter than rope A, so then we say that it is $\frac{3}{4}$ times as long as rope A. They see that it is possible that the answer to a “how many groups?” question is a number less than 1 when the given amount is smaller than the size of a group.

As students work, notice how they go about making multiplicative comparisons. Select students who write clear and concise questions for the equations in the last problem so they can share later.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR8: Discussion Supports
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 4–5 minutes of quiet think time and then a couple of minutes to compare their responses with a partner and discuss any disagreements. Clarify that rope A is 4 units long.

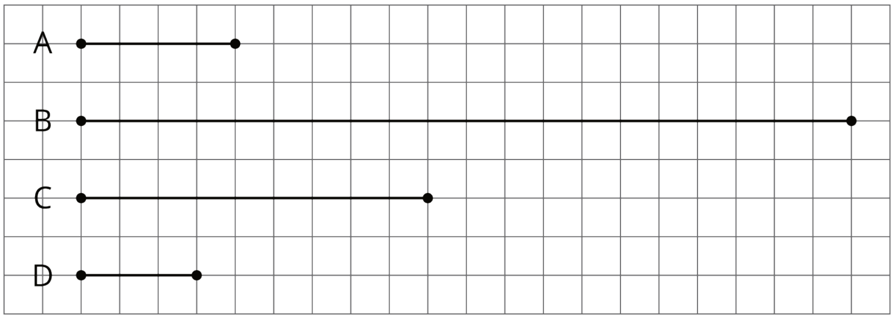
Students using the digital materials can use the applet at ggbm.at/kZUgANCC to compare the segments. The segments can be moved by dragging endpoints with open circles. The yellow “pins” can help students keep track of the groups.

Anticipated Misconceptions

Some students might associate the wrong lengths with the ropes or confuse the order of comparison (e.g., comparing A to C instead of C to A). Encourage them to put the length of each rope next to the diagram and attend more closely to the ropes being compared.

Student Task Statement

Here is a diagram that shows four ropes of different lengths.



1. Complete each sentence comparing the lengths of the ropes. Then, use the measurements shown on the grid to write a multiplication equation and a division equation for each comparison.

    a. Rope B is _______ times as long as Rope A.

    b. Rope C is _______ times as long as Rope A.

    c. Rope D is _______ times as long as Rope A.

2. Each equation can be used to answer a question about Ropes C and D. What could each question be?

    a. $? \cdot 3 = 9$ and $9 \div 3 = ?$

    b. $? \cdot 9 = 3$ and $3 \div 9 = ?$

Student Response

1. a. Rope B is 5 times as long as Rope A. $5 \cdot 4 = 20$ (or $4 \cdot 5 = 20$) and $20 \div 4 = 5$ (or $20 \div 5 = 4$)

    b. Rope C is $2\frac{1}{4}$ (or equivalent) times as long as Rope A. $2\frac{1}{4} \cdot 4 = 9$ (or $4 \cdot 2\frac{1}{4} = 9$) and $9 \div 4 = 2\frac{1}{4}$ (or $9 \div 2\frac{1}{4} = 4$)

    c. Rope D is $\frac{3}{4}$ (or equivalent) times as long as Rope A. $\frac{3}{4} \cdot 4 = 3$ (or $4 \cdot \frac{3}{4} = 3$) and $3 \div 4 = \frac{3}{4}$ (or $3 \div \frac{3}{4} = 4$)

2. a. Responses vary. Sample response: How many times as long as Rope D is Rope C? (or how many times does the length of Rope D go into that of Rope C?)

    b. Responses vary. Sample response: How many times as long as Rope C is Rope D? (or how many times does the length of Rope C go into that of Rope D?)

Activity Synthesis

Display the solutions to the first set of problems for all to see. Give students a minute to check their answers and ask questions. Then, focus class discussion on two key ideas:

- The connection between "how many groups?" questions and "how many times as long?" questions. Ask students how these two types of questions are similar and different. Make sure students see that both have the structure of $? \cdot a = b$, where a is the size of 1 group (or the unit we are using for comparison), and b is a given number.
- The language commonly used when referring to a situation in which the number of groups is less than 1 whole. Explain that we have seen equal-sized groups where the number of groups is greater than 1, but some situations involve a part of 1 group. So instead of saying "the number of groups" or asking "how many groups?," we would ask "what fraction of a group?" or "what part of a group?". For example, in the case of rope D, where the answer is less than 1, we can ask, "What fraction of rope A is rope D?"

Ask 1–2 previously identified students to share the question they wrote for the last pair of equations ($? \cdot 9 = 3$ and $3 \div 9 = ?$). Make sure students see that this pair of equations represent a situation with a fractional group
(i.e., rope D is shorter than rope C, so the length of rope D is a fraction of that of rope C).

Access for Students with Disabilities

Representation: Internalize Comprehension. Use color and annotations to illustrate connections between representations. As students describe their reasoning about "how many groups?" and "how many times as long?", use color and annotations to scribe their thinking on a display of each problem so that it is visible for all students.
Supports accessibility for: Visual-spatial processing; Conceptual processing

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Access for English Language Learners

Speaking: MLR8 Discussion Supports. As students compare and contrast these two types of division questions, provide a sentence frame such as: "Something these two types of questions have in common is . . ." and "A difference between these two types of questions is . . . ." This will help students produce and make sense of the language needed to communicate their ideas about the relationship between multiplication and division equations.
Design Principle(s): Support sense-making; Optimize output (for comparison)

7.3 Fractional Batches of Ice Cream

20 minutes

In this activity, students make sense of quotients that are less than 1 and greater than 1 in the same context. Later in the task, students generalize their reasoning to solve division problems (where the quotient is less than 1) without contexts.

Given the amount of milk required for 1 batch of ice cream (i.e., the size of 1 group), students find out how many batches (i.e., the number of groups or what fraction of a group) can be made with different amounts of milk. They continue to use tape diagrams and write equations to reason about the situations, but this time, they are not prompted to write multiplication equations.

As students work, identify students students who drew clear and effective diagrams for the ice cream problems. Select them to share later.

Addressing

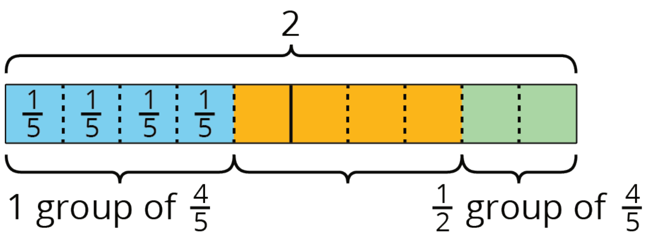
- 6.NS.A.1

Instructional Routines

- MLR6: Three Reads
- Think Pair Share

Launch

Keep students in groups of 2. Display an example of a tape diagram that students have used in a previous lesson. The diagram for the question “how many $\frac{4}{5}$s are in 2?” from a previous cool-down is shown here.



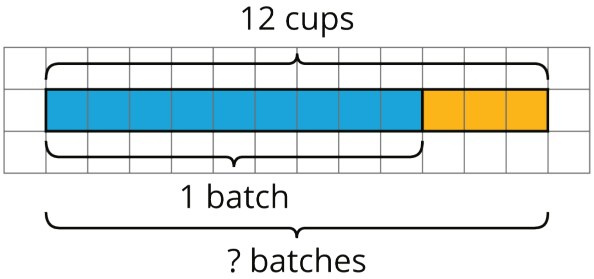
Point out how the diagram shows both full groups of $\frac{4}{5}$ and a partial group. Tell students that they will see more situations involving partial groups in this activity.

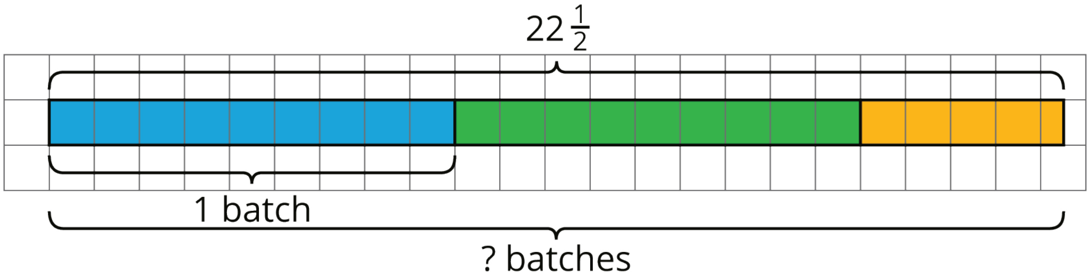
Give students 6–8 minutes of quiet work time for the first two sets of questions about ice cream. Ask students to make a quick estimate on whether each answer will be greater than or less than 1 before solving the problem. Provide access to colored pencils, as some students may find it helpful to identify whole groups and partial groups on a tape diagram by coloring.

Give students 2–3 minutes to discuss their responses with their partner. Follow with a whole-class discussion before students return to the last set of questions.

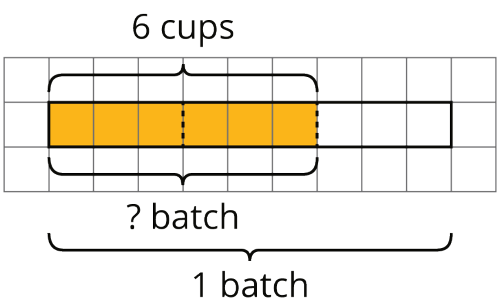
Ask previously identified students to share their diagrams for the ice cream problems, or display the following diagrams.

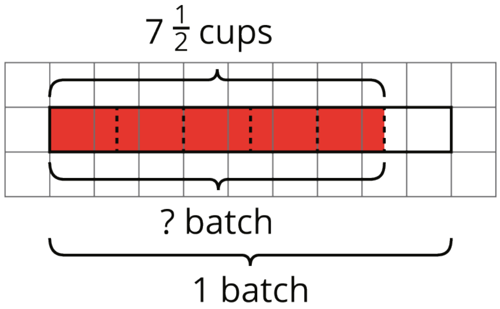
Monday and Tuesday:





Thursday and Friday:





To help students see the structure in the diagrams, ask: “How are the diagrams for Monday and Tuesday like and unlike those for Thursday and Friday?” If not brought up in their responses, point out that:

- The size of 1 group (or the amount of milk in 1 batch) is the same in all diagrams, but the amounts we are comparing to 1 group vary. Those amounts are greater than 1 batch (9 cups) on Monday and Tuesday, and less than 1 batch on Thursday and Friday.
- This comparison to the size of 1 group is also reflected in the questions. We ask “how many batches?” for the first two, and “what fraction of a batch?” for the other two.

To help students notice the structure in the equations, ask: “How are the division equations for Monday and Tuesday different than those for Thursday and Friday? How are they the same?”

$$12 \div 9 = ?$$

$$22\frac{1}{2} \div 9 = ?$$

$$6 \div 9 = ?$$

$$7\frac{1}{2} \div 9 = ?$$

Highlight that, regardless of whether the answer is greater than 1 or less than 1, the equations show that the questions "how many batches (of 9 cups)?" and "what fraction of a batch (of 9 cups)?" can be expressed with a division by 9, because the multiplication counterparts of these situations all have the structure of "what number times 9 equals a given amount of milk?" or

$$? \cdot 9 = b$$

where b is a given amount of milk.

Give students quiet time to complete the last set of questions.

Access for English Language Learners

Reading: MLR6 Three Reads. Use this routine to support reading comprehension of this word problem, without solving it for students. Use the first read to orient students to the situation. After a shared reading, ask students "what is this situation about?" (A chef makes makes different amounts of ice cream on different days). After the second read, students list any quantities that can be counted or measured, without focusing on specific values (number of cups of milk needed for every batch of ice cream, number of cups of milk used each day). Listen for, and amplify, the two important quantities that vary in relation to each other in this situation: number of cups of milk, and number of (or part of) batches of ice cream. After the third read, ask students to brainstorm possible strategies to answer the question.
Design Principle(s): Support sense-making

Anticipated Misconceptions

If students are not sure how to begin representing a situation with a tape diagram, ask them to represent one quantity or number at a time. For example, they could begin by showing the amount of milk used as a tape with a particular length, and then mark the second quantity (the amount of milk in 1 batch) on the same tape and with the same starting point. Or they could represent the amounts in the opposite order.

Student Task Statement

One batch of an ice cream recipe uses 9 cups of milk. A chef makes different amounts of ice cream on different days. Here are the amounts of milk she used:

- Monday: 12 cups
- Tuesday: $22\frac{1}{2}$ cups
- Thursday: 6 cups
- Friday: $7\frac{1}{2}$ cups

1. How many batches of ice cream did she make on these days? For each day, write a division equation, draw a tape diagram, and find the answer.

    a. Monday

b. Tuesday

2. What fraction of a batch of ice cream did she make on these days? For each day, write a division equation, draw a tape diagram, and find the answer.

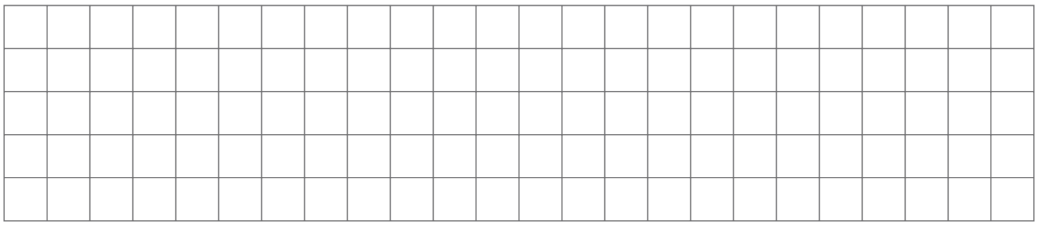
    a. Thursday

    b. Friday

3. For each question, write a division equation, draw a tape diagram, and find the answer.

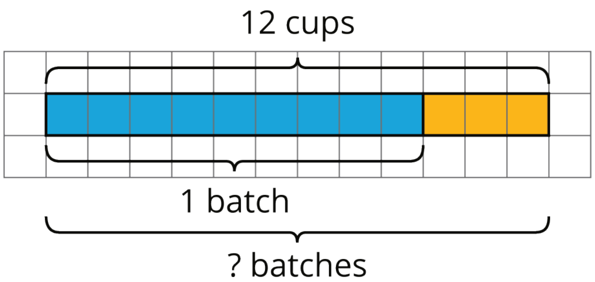
    a. What fraction of 9 is 3?

    b. What fraction of 5 is $\frac{1}{2}$?

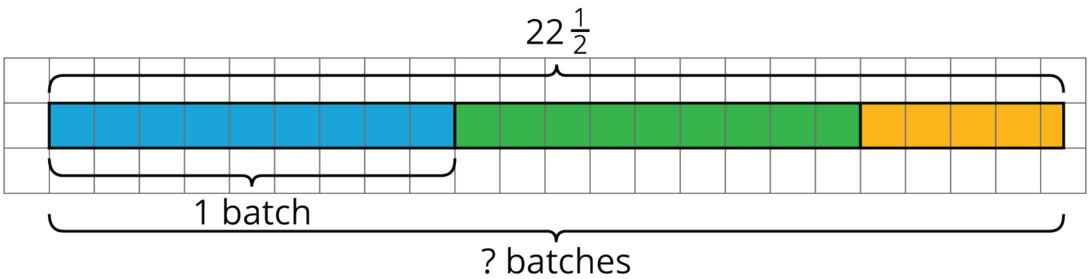


Student Response

1. a. Equation: $12 \div 9 = ?$ (or $12 \div ? = 9$). Solution: $1\frac{1}{3}$ batches (or equivalent).

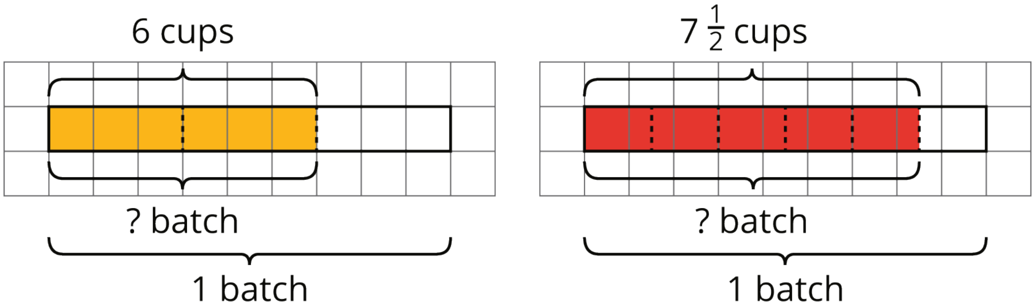


   b. Equation: $22\frac{1}{2} \div 9 = ?$ (or $22\frac{1}{2} \div ? = 9$). Solution: $2\frac{1}{2}$ batches (or equivalent).



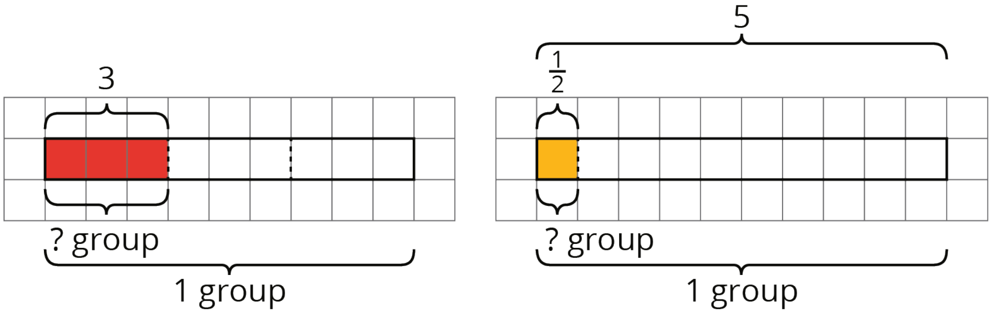
2. a. Equation: $6 \div 9 = ?$ (or $6 \div ? = 9$). Solution: $\frac{2}{3}$ of a batch (or equivalent).

   b. Equation: $7\frac{1}{2} \div 9 = ?$ (or $7\frac{1}{2} \div ? = 9$). Solution: $\frac{5}{6}$ of a batch (or equivalent).



3. a. Equation: $3 \div 9 = ?$ (or $3 \div ? = 9$). Solution: $\frac{1}{3}$.

   b. Equation: $\frac{1}{2} \div 5 = ?$ (or $\frac{1}{2} \div ? = 5$). Solution: $\frac{1}{10}$ (or 0.1).

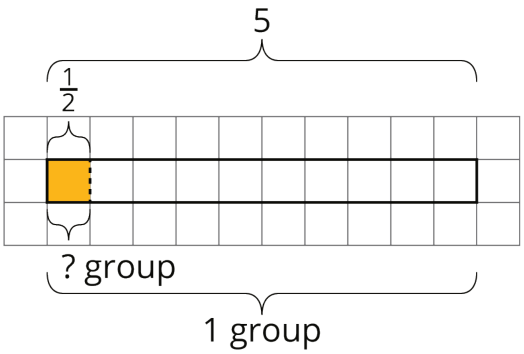


Activity Synthesis

After students worked on the last set of problems, discuss the question "what fraction of 5 is $\frac{1}{2}$?" To help students connect this question to previous ones, consider asking:

- "How can we tell if the answer is greater than 1 or less than 1 before calculating?" (The phrase "what fraction of" offers a clue that it is less than 1. Or, we are comparing $\frac{1}{2}$ to 5 and can see that $\frac{1}{2}$ is less than 5.)
- "What is the size of 1 group here? How do we know?" (We can tell that 5 is the size of 1 group because that is the value to which another number is being compared.)
- "How do we write a multiplication equation for this question? A division equation?" ($? \cdot 5 = \frac{1}{2}$, and $\frac{1}{2} \div 5 = ?$)

Select a student to display a correct diagram for the problem, or display this diagram for all to see. Discuss how the two given values and the solution are represented in the diagram.



Lesson Synthesis

In this lesson, we saw that a division problem can represent the idea of equal-sized groups but may have a total amount that is less than the size of one full group. Instead of "how many of this is in that?", the question is now "what fraction of this is that?".

- "How can we tell if a division situation involves less than one whole group?" (The total amount is less than the size of the a group, or the question asks "what fraction of. . .?")

- "How do we find quotients that are less than 1?" (We can write a multiplication equation that corresponds to the situation and draw a tape diagram to help us reason about what fraction of 1 group the given amount is.)

We also explored division problems as representing the answers to comparison questions about measurement. Instead of "how many groups?", we can ask "how many times as long (or as heavy)?" For example: $16 \div 4 = ?$ corresponds to $? \cdot 4 = 16$, which can represent the question "how many times as long as 4 cm is 16 cm?" We can reason that 16 cm is 4 times as long as 4 cm.

In the same context, $3 \div 4$ corresponds to $? \cdot 4 = 3$, and would mean "how many times as long as 4 cm is 3 cm?" Here, we can see that the answer will be a fraction less than 1. Because 3 is $\frac{3}{4}$ of 4, we can say "3 cm is $\frac{3}{4}$ as long as 4 cm."

7.4 A Partially Filled Container

Cool Down: 5 minutes

Addressing

- 6.NS.A.1

Launch

Provide continued access to colored pencils.

Student Task Statement

There is $\frac{1}{3}$ gallon of water in a 3-gallon container. What fraction of the container is filled?

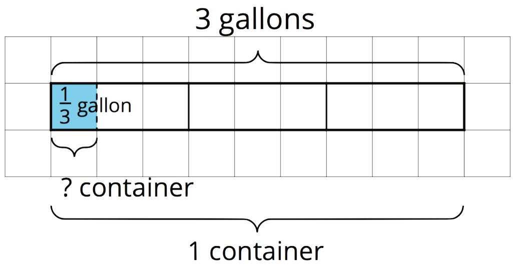
1. Write a multiplication equation and a division equation to represent the situation.

2. Draw a tape diagram to represent the situation. Then, answer the question.

Student Response

1. $? \cdot 3 = \frac{1}{3}$ (or $3 \cdot ? = \frac{1}{3}$) and $\frac{1}{3} \div 3 = ?$ (or $\frac{1}{3} \div ? = 3$)

2. Diagrams vary. Sample diagram:

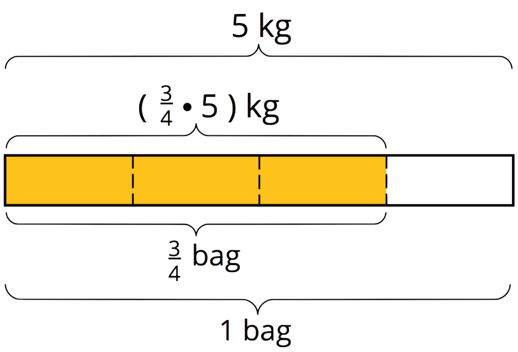


$\frac{1}{9}$ of the container is filled.

Student Lesson Summary

It is natural to think about groups when we have more than one group, but we can also have a *fraction of a group*.

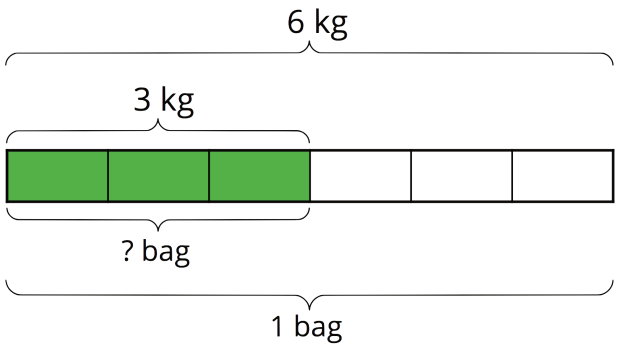
To find the amount in a fraction of a group, we can multiply the fraction by the amount in the whole group. If a bag of rice weighs 5 kg, $\frac{3}{4}$ of a bag would weigh $(\frac{3}{4} \cdot 5)$ kg.



Sometimes we need to find what fraction of a group an amount is. Suppose a full bag of flour weighs 6 kg. A chef used 3 kg of flour. What fraction of a full bag was used? In other words, what fraction of 6 kg is 3 kg?

This question can be represented by a multiplication equation and a division equation, as well as by a diagram.

$$? \cdot 6 = 3$$
$$3 \div 6 = ?$$

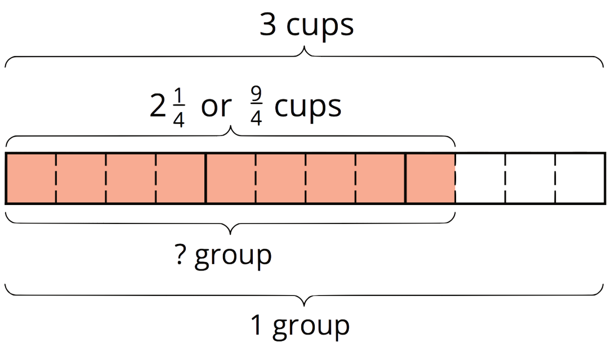


We can see from the diagram that 3 is $\frac{1}{2}$ of 6, and we can check this answer by multiplying: $\frac{1}{2} \cdot 6 = 3$.

In *any* situation where we want to know what fraction one number is of another number, we can write a division equation to help us find the answer.

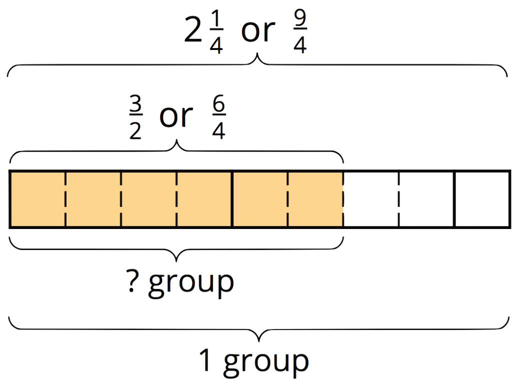
For example, "What fraction of 3 is $2\frac{1}{4}$?" can be expressed as $? \cdot 3 = 2\frac{1}{4}$, which can also be written as $2\frac{1}{4} \div 3 = ?$.

The answer to "What is $2\frac{1}{4} \div 3$?" is also the answer to the original question.



The diagram shows that 3 wholes contain 12 fourths, and $2\frac{1}{4}$ contains 9 fourths, so the answer to this question is $\frac{9}{12}$, which is equivalent to $\frac{3}{4}$.

We can use diagrams to help us solve other division problems that require finding a fraction of a group. For example, here is a diagram to help us answer the question: "What fraction of $\frac{9}{4}$ is $\frac{3}{2}$?," which can be written as $\frac{3}{2} \div \frac{9}{4} = ?$.



We can see that the quotient is $\frac{6}{9}$, which is equivalent to $\frac{2}{3}$. To check this, let's multiply. $\frac{2}{3} \cdot \frac{9}{4} = \frac{18}{12}$, and $\frac{18}{12}$ is, indeed, equal to $\frac{3}{2}$.

Lesson 7 Practice Problems

Problem 1

Statement

A recipe calls for $\frac{1}{2}$ lb of flour for 1 batch. How many batches can be made with each of these amounts?

a. 1 lb

b. $\frac{3}{4}$ lb

c. $\frac{1}{4}$ lb

Solution

a. 2

b. $1\frac{1}{2}$

c. $\frac{1}{2}$

Problem 2

Statement

Whiskers the cat weighs $2\frac{2}{3}$ kg. Piglio weighs 4 kg. For each question, write a multiplication equation and a division equation, decide whether the answer is greater than 1 or less than 1, and then find the answer.

a. How many times as heavy as Piglio is Whiskers?

b. How many times as heavy as Whiskers is Piglio?

Solution

a. Multiplication: $? \cdot 4 = 2\frac{2}{3}$ (or $4 \cdot ? = 2\frac{2}{3}$), division: $\left(2\frac{2}{3}\right) \div 4 = ?$. Less than 1. Cat A is $\frac{8}{12}$ (or $\frac{2}{3}$) as heavy as Cat B.

b. Multiplication: $? \cdot \left(2\frac{2}{3}\right) = 4$ (or $\left(2\frac{2}{3}\right) \cdot ? = 4$), division: $4 \div \left(2\frac{2}{3}\right) = ?$. Bigger than 1. Cat B is $1\frac{4}{8}$ or $1\frac{1}{2}$ times as heavy Cat A.

Problem 3

Statement

Andre is walking from his home to a festival that is $1\frac{5}{8}$ kilometers away. He walks $\frac{1}{3}$ kilometer and then takes a quick rest. Which question can be represented by the equation $? \cdot 1\frac{5}{8} = \frac{1}{3}$ in this situation?

A. What fraction of the trip has Andre completed?

B. What fraction of the trip is left?

C. How many more kilometers does Andre have to walk to get to the festival?

D. How many kilometers is it from home to the festival and back home?

Solution

A

Problem 4

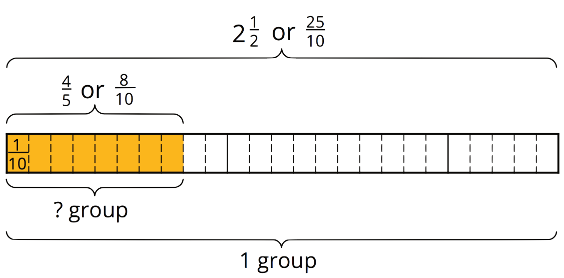
Statement

Draw a tape diagram to represent the question: What fraction of $2\frac{1}{2}$ is $\frac{4}{5}$?

Then find the answer.

Solution

$\frac{8}{25}$. Sample diagram:



Problem 5

Statement

How many groups of $\frac{3}{4}$ are in each of these quantities?

a. $\frac{11}{4}$

b. $6\frac{1}{2}$

Solution

a. $3\frac{2}{3}$ Sample reasoning: create a tape diagram showing $\frac{11}{4}$ divided into groups of $\frac{3}{4}$ each.

b. $8\frac{2}{3}$ Sample reasoning: create a tape diagram showing $6\frac{1}{2}$ divided into groups of $\frac{3}{4}$ each.

(From Unit 4, Lesson 6.)

Problem 6

Statement

Which question can be represented by the equation $4 \div \frac{2}{7} = ?$

A. What is 4 groups of $\frac{2}{7}$?

B. How many $\frac{2}{7}$s are in 4?

C. What is $\frac{2}{7}$ of 4?

D. How many 4s are in $\frac{2}{7}$?

Solution

B

(From Unit 4, Lesson 4.)