Lesson 6: What about Other Bases?

Goals

- Generalize exponent rules for nonzero bases, including bases other than 10.
- Use exponent rules to identify (in writing) equivalent exponential expressions, and explain (orally) the reasoning.

Learning Targets

- I can use the exponent rules for bases other than 10.

Lesson Narrative

In this lesson, students use their understanding of exponent rules with powers of 10 to make sense of exponent rules for powers of other bases. Students work through problems analogous to the ones used to develop exponent rules for powers of 10. The same underlying patterns emerge, revealing the fact that the exponent rules are the same for bases other than 10. For simplicity, students do not develop general exponent rules for negative bases.

Alignments

Addressing

- 8.EE.A.1: Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.

Instructional Routines

- MLR2: Collect and Display
- MLR8: Discussion Supports
- Think Pair Share
- True or False

Required Materials

Tools for creating a visual display
Any way for students to create work that can be easily displayed to the class. Examples: chart paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

Required Preparation

Create a new set of visual displays for exponent rules that replace the base of 10 with other bases. For example, the visual display for the rule $10^n \cdot 10^m = 10^{n+m}$ should be updated to a new visual display with $b^n \cdot b^m = b^{n+m}$ with a guiding example that uses a base other than 10.

Student Learning Goals

Let's explore exponent patterns with bases other than 10.

6.1 True or False: Comparing Expressions with Exponents

Warm Up: 5 minutes

The purpose of this warm-up is for students to reason about powers of positive and negative numbers. While some students may find the numeric value of each expression to make a comparison, it is not always necessary. Encourage students to reason about the meaning of the integers, bases, and exponents rather than finding the numeric value of each expression.

Addressing

- 8.EE.A.1

Instructional Routines

- True or False

Launch

Display one problem at a time. Give students up to 1 minute of quiet think time per problem and tell them to give a signal when they have an answer and a strategy. Follow with a whole-class discussion.

There may be some confusion about how negative numbers interact with exponents. Tell students that exponents work the same way for both positive and negative bases. In other words, $(-3)^2 = -3 \cdot -3$.

Students may also struggle to interpret the notation difference between two expressions such as $(-4)^2$ and -4^2. Tell students that the interpretation of the first expression is $(-4)^2 = -4 \cdot -4$ and the interpretation of the second expression is $-4^2 = -1 \cdot 4 \cdot 4$.

Student Task Statement

Is each statement true or false? Be prepared to explain your reasoning.

1. $3^5 < 4^6$
2. $(-3)^2 < 3^2$
3. $(-3)^3 = 3^3$
4. $(-5)^2 > -5^2$

Student Response

1. True. Sample response: The base and exponent are both smaller on the left.
2. False. Sample response: When you square a negative, the result is positive.

3. False. Sample response: When you cube a negative, the result is negative, and a negative is always less than a positive.
4. True. Sample response: A positive is always greater than a negative.

Activity Synthesis

Poll students on their response for each problem. Record and display their responses for all to see. If all students agree, ask 1 or 2 students to share their reasoning. If there is disagreement, ask students to share their reasoning until an agreement is reached. After the final problem, ask students if the same strategies would have worked if all of the bases were 10 to highlight the idea that powers with a base of 10 behave the same as those with bases other than 10.

To involve more students in the conversation, consider asking:

- "Do you agree or disagree? Why?"
- "Who can restate ___'s reasoning in a different way?"
- "Does anyone want to add on to _____'s reasoning?"

6.2 What Happens with Zero and Negative Exponents?

15 minutes

Extend the definition of non-positive exponents to other bases. The work of this activity is parallel to the work with the powers of 10 table in the previous lesson. Notice students who transfer their understanding of powers of 10, especially non-positive exponents, to exponential expressions with bases other than 10.

Addressing

- 8.EE.A.1

Instructional Routines

- MLR2: Collect and Display
- Think Pair Share

Launch

Arrange students in groups of 2. Tell students to work on their own and then share their reasoning with their partner after each has had a chance to complete the problem. Encourage students to look for similarities to work they have already done with base 10. Give students 10 minutes of work time, followed by a whole class discussion.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Check in with students within the first 2-3 minutes of work time. Look for students who did not notice direction of the arrows on the table.
Supports accessibility for: Visual-spatial processing; Organization

Access for English Language Learners

Conversing, Representing, Writing: MLR2 Collect and Display. Use this routine to support small-group and whole-class discussion. Circulate and listen to students as they work, capture the vocabulary and phrases students use to describe the patterns they noticed and how these are related to exponent rules. Listen for students who make connections between repeated multiplication, reciprocals, and negative and positive exponents. Record their language and any relevant diagrams onto a visual display that can be referenced in future discussions. This will help students produce and make sense of the language needed to communicate their reasoning of exponent rules for powers of positive bases other than 10.
Design Principle(s): Support sense-making; Maximize meta-awareness

Student Task Statement

Complete the table to show what it means to have an exponent of zero or a negative exponent.

·2 ·2 ·2 ·2 ·2 ·2 ·2 ·2

value	16					$\frac{1}{2}$			
exponent form	2^4								

· ? · ? · ? · ? · ? · ? · ? · ?

1. As you move toward the left, each number is being multiplied by 2. What is the multiplier as you move toward the right?
2. Use the patterns you found in the table to write 2^{-6} as a fraction.
3. Write $\frac{1}{32}$ as a power of 2 with a single exponent.
4. What is the value of 2^0?
5. From the work you have done with negative exponents, how would you write 5^{-3} as a fraction?
6. How would you write 3^{-4} as a fraction?

Student Response

value	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
exponent form	2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3}	2^{-4}

1. As you move right, each number is multiplied by $\frac{1}{2}$.
2. $2^{-6} = \frac{1}{2^6}$ or $\frac{1}{64}$
3. $\frac{1}{32} = \frac{1}{2^5} = 2^{-5}$
4. $2^0 = 1.$
5. $5^{-3} = \frac{1}{5^3}$ or $\frac{1}{125}$
6. $3^{-4} = \frac{1}{3^4}$ or $\frac{1}{81}$

Are You Ready for More?

1. Find an expression equivalent to $\left(\frac{2}{3}\right)^{-3}$ but with positive exponents.
2. Find an expression equivalent to $\left(\frac{4}{5}\right)^{-8}$ but with positive exponents.
3. What patterns do you notice when you start with a fraction raised to a negative exponent and rewrite it using a single positive exponent? Show or explain your reasoning.

Student Response

1. $\left(\frac{3}{2}\right)^3$, because the opposite of repeatedly multiplying by $\frac{2}{3}$ is repeatedly multiplying by $\frac{3}{2}$.
2. $\left(\frac{5}{4}\right)^8$, because the opposite of repeatedly multiplying by $\frac{4}{5}$ is repeatedly multiplying by $\frac{5}{4}$.
3. To write $\left(\frac{a}{b}\right)^{-n}$ with a positive exponent, notice that the opposite of repeatedly multiplying by $\frac{a}{b}$ is repeatedly multiplying by $\frac{b}{a}$. The general rule is that a fraction to a negative exponent is equal to the reciprocal to the positive exponent. So $\left(\frac{b}{a}\right)^n$

Activity Synthesis

The important idea for this activity is that the same logic that we used to analyze powers of 10 applies to other bases as well. Students will work mostly with bases that are whole numbers and positive fractions.

Here are some sample questions for whole-class discussion:

- "How does working with powers of 2 compare to working with powers of 10?" (The properties apply in the same way, but computing the value of the exponents is more difficult.)
- "What happens when you use exponent rules when one of the exponents is 0?" (The rules apply in the same way. $2^0 = 1$ in the same way that $10^0 = 1$.)
- "How does 10^{-3} compare to 2^{-3}?" (Both values are positive and less than 1. Since 10^{-3} has greater numbers in the denominator, it is less than 2^{-3}.)

Introduce new (or update current) visual displays to illustrate exponent rules for positive, rational bases. Display them for all to see throughout the unit. An example would be to update the rule $10^n \cdot 10^m = 10^{n+m}$ with $x^n \cdot x^m = x^{n+m}$ with a guiding example that uses a base other than 10.

6.3 Exponent Rules with Bases Other than 10

15 minutes

This activity gives students a chance to practice using exponent rules to analyze expressions and identify equivalent ones. Five lists of expressions are given. Students choose two lists to analyze. As students work, notice the different strategies used to analyze the expressions in each list. Ask students using contrasting strategies to share later.

Addressing

- 8.EE.A.1

Instructional Routines

- MLR8: Discussion Supports

Launch

Arrange students in groups of 2. Tell students to take turns selecting an expression, determining whether the expression matches the original, and explaining their reasoning. Give students 10 minutes to work followed by a whole-class discussion.

Anticipated Misconceptions

Some students may struggle with comparing -5^9 to 5^{-9}. Ask these students to explain the difference between 10^3 and 10^{-3}, and relate that to the difference between 5^9 and 5^{-9}. If needed, explain that -5^9 is a large negative number, whereas 5^{-9} is a very small positive number.

Student Task Statement

Lin, Noah, Diego, and Elena decide to test each other's knowledge of exponents with bases other than 10. They each chose an expression to start with and then came up with a new list of expressions; some of which are equivalent to the original and some of which are not.

Choose 2 of the 4 lists to analyze. For each list of expressions you choose to analyze, decide which expressions are *not* equivalent to the original. Be prepared to explain your reasoning.

1. Lin's original expression is 5^{-9} and her list is:

$(5^3)^{-3}$ -5^9 $\frac{5^{-6}}{5^3}$ $(5^3)^{-2}$ $\frac{5^{-4}}{5^{-5}}$ $5^{-4} \cdot 5^{-5}$

2. Noah's original expression is 3^{10} and his list is:

$3^5 \cdot 3^2$ $(3^5)^2$ $(3 \cdot 3)(3 \cdot 3)(3 \cdot 3)(3 \cdot 3)(3 \cdot 3)$

$\left(\frac{1}{3}\right)^{-10}$ $3^7 \cdot 3^3$ $\frac{3^{20}}{3^{10}}$ $\frac{3^{20}}{3^2}$

3. Diego's original expression is x^4 and his list is:

$\frac{x^8}{x^4}$ $x \cdot x \cdot x \cdot x$ $\frac{x^{-4}}{x^{-8}}$ $\frac{x^{-4}}{x^8}$

$(x^2)^2$ $4 \cdot x$ $x \cdot x^3$

4. Elena's original expression is 8^0 and her list is:

1 0 $8^3 \cdot 8^{-3}$

$\frac{8^2}{8^2}$ 10^0 11^0

Student Response

1. The following are not equivalent to 5^{-9}:
   - -5^9 because it is negative and 5^{-9} is positive.
   - $\left(5^3\right)^{-2}$ because it is equal to 5^{-6}.
   - $\frac{5^{-4}}{5^{-5}}$ because $\frac{5^{-4}}{5^{-5}} = 5^{-4-(-5)} = 5^{-4+5} = 5^1$.
2. The following are not equivalent to 3^{10}:
   - $3^5 \cdot 3^2$ because $3^5 \cdot 3^2 = 3^{5+2} = 3^7$.
   - $\frac{3^{20}}{3^2}$ because $\frac{3^{20}}{3^2} = 3^{20-2} = 3^{18}$.
3. The following are not equivalent to x^4:
   - $\frac{x^{-4}}{x^8}$ because $\frac{x^{-4}}{x^8} = x^{-4-8} = x^{-12}$.
   - $4 \cdot x$. Sample reasoning: If $x = 10$, then $4 \cdot 10 = 40$ and $10^4 = 10000$.
4. The only expression not equivalent to 8^0 in Elena's list is 0, because all of the rest are defined to be equal to 1.

Activity Synthesis

The key idea is that the exponent rules are valid for nonzero bases other than 10. Select students to share which expressions did not match the original in the lists they chose to study. Consider focusing on questions 3 and 4 to assess how students have generalized in terms of variables and whether students have internalized the 0 exponent. Select previously identified students to share the different strategies used to analyze the same list.

Here are some questions to consider for discussion:

- "What mistakes might lead to the expressions that are not equivalent to the original expression?"
- "How do you know that expression is not equivalent to the original?"
- "Did anyone think about that expression in a different way?"

Access for English Language Learners

Speaking: MLR8 Discussion Supports. To support students in explaining why identified expressions are or are not equivalent, provide sentence frames for students to use when they are working with their partner. For example, "____ and _____ are (equivalent/not equivalent), because _____ ." or "I (agree/disagree) because _____."
Design Principle(s): Support sense-making; Optimize output for (explanation)

Lesson Synthesis

The purpose of the discussion is to check whether students understand that the rules they have developed for doing arithmetic with powers of 10 are valid for any other non-zero rational bases as well. This is also a good opportunity to address possible misconceptions students have, especially regarding non-positive exponents.

Consider using these discussion questions to emphasize the important concepts in this lesson:

- "We saw before that 10^3 shows the repeated multiplication $10 \cdot 10 \cdot 10$ and 10^{-3} shows the repeated multiplication $\frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10}$. How do 2^3 and 2^{-3} compare to 10^3 and 10^{-3}?" (When the base is 2 rather than 10, the repeated multiplication works the same way. 2^3 is $2 \cdot 2 \cdot 2$ and 2^{-3} is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$.)
- "How do 5^3 and 5^{-3} compare to 10^3 and 10^{-3}?" (The exponent rules work in exactly the same way. The base is just 5 instead of 10.)
- "How is $(-3)^4$ different than -3^4? If you get stuck, think about what base is being repeatedly multiplied." ($(-3)^4$ is $(-3)(-3)(-3)(-3)$ whereas -3^4 is $-(3 \cdot 3 \cdot 3 \cdot 3)$.)

- "Do the other exponent rules work the same way that they did for base 10? Give some examples with different bases." (Yes, the other exponents rules work the same way. With base 7 for example, we have that $\frac{7^5}{7^2} = 7^3$ and $7^3 \cdot 7^6 = 7^9$.)
- "What about combining bases? For example, do you agree that $3^5 \cdot 4^3 = 12^8$?" (No, you cannot combine bases in this way. To see this, expand the factors of both sides. The left side has 5 factors that are 3 and 3 factors that are 4. One could combine some of these factors to make 12, but the right side has 8 factors that are 12. They are not equal.)

Close by telling students that the exponent rules with other bases work exactly the same way as they did with 10 and they can always check this by expanding the factors of an expression with exponents (unless the exponent is very large and it would take too long to expand).

6.4 Spot the Mistake

Cool Down: 5 minutes
Students examine common conceptual errors to reveal their current understanding.

Addressing

- 8.EE.A.1

Student Task Statement

1. Diego was trying to write $2^3 \cdot 2^2$ with a single exponent and wrote $2^3 \cdot 2^2 = 2^{3 \cdot 2} = 2^6$. Explain to Diego what his mistake was and what the answer should be.
2. Andre was trying to write $\frac{7^4}{7^{-3}}$ with a single exponent and wrote $\frac{7^4}{7^{-3}} = 7^{4-3} = 7^1$. Explain to Andre what his mistake was and what the answer should be.

Student Response

1. 2^5. Diego multiplied the exponents when he should have added them. To see this, he could have expanded the expressions: $2^3 \cdot 2^2 = (2 \cdot 2 \cdot 2)(2 \cdot 2) = 2^{3+2} = 2^5$.
2. 7^7. Andre did 7^{4-3} when he should have done $7^{4-(-3)}$.

Student Lesson Summary

Earlier we focused on powers of 10 because 10 plays a special role in the decimal number system. But the exponent rules that we developed for 10 also work for other bases. For example, if $2^0 = 1$ and $2^{-n} = \frac{1}{2^n}$, then

$$2^m \cdot 2^n = 2^{m+n}$$
$$(2^m)^n = 2^{m \cdot n}$$
$$\frac{2^m}{2^n} = 2^{m-n}.$$

These rules also work for powers of numbers less than 1. For example, $\left(\frac{1}{3}\right)^2 = \frac{1}{3} \cdot \frac{1}{3}$ and $\left(\frac{1}{3}\right)^4 = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$. We can also check that $\left(\frac{1}{3}\right)^2 \cdot \left(\frac{1}{3}\right)^4 = \left(\frac{1}{3}\right)^{2+4}$.

Using a variable x helps to see this structure. Since $x^2 \cdot x^5 = x^7$ (both sides have 7 factors that are x), if we let $x = 4$, we can see that $4^2 \cdot 4^5 = 4^7$. Similarly, we could let $x = \frac{2}{3}$ or $x = 11$ or any other positive value and show that these relationships still hold.

Lesson 6 Practice Problems

Problem 1

Statement

Priya says "I can figure out 5^0 by looking at other powers of 5. 5^3 is 125, 5^2 is 25, then 5^1 is 5."

a. What pattern do you notice?

b. If this pattern continues, what should be the value of 5^0? Explain how you know.

c. If this pattern continues, what should be the value of 5^{-1}? Explain how you know.

Solution

a. When the power of 5 drops by one, the value is divided by 5.

b. 1. The value of 5^0 should be the value of 5^1 divided by 5.

c. $\frac{1}{5}$. The value of 5^{-1} should be the value of 5^0 divided by 5.

Problem 2

Statement

Select **all** the expressions that are equivalent to 4^{-3}.

A. -12

B. 2^{-6}

C. $\frac{1}{4^3}$

D. $\left(\frac{1}{4}\right) \cdot \left(\frac{1}{4}\right) \cdot \left(\frac{1}{4}\right)$

E. 12

F. $(-4) \cdot (-4) \cdot (-4)$

G. $\frac{8^{-1}}{2^2}$

Solution

["B", "C", "D"]

Problem 3

Statement

Write each expression using a single exponent.

a. $\frac{5^3}{5^6}$

b. $(14^3)^6$

c. $8^3 \cdot 8^6$

d. $\frac{16^6}{16^3}$

e. $(21^3)^{-6}$

Solution

a. 5^{-3}

b. 14^{18}

c. 8^9

d. 16^3

e. 21^{-18}

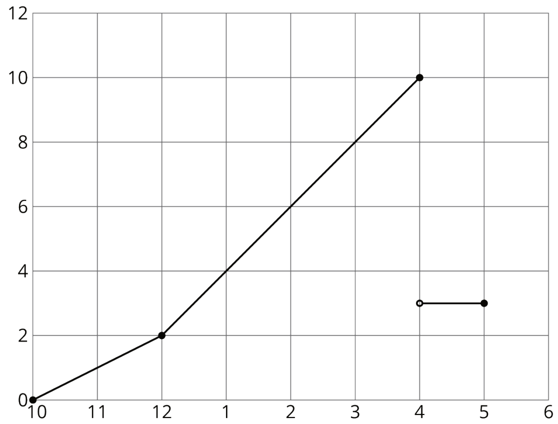
Problem 4

Statement

Andre sets up a rain gauge to measure rainfall in his back yard. On Tuesday, it rains off and on all day.

- He starts at 10 a.m. with an empty gauge when it starts to rain.
- Two hours later, he checks, and the gauge has 2 cm of water in it.
- It starts raining even harder, and at 4 p.m., the rain stops, so Andre checks the rain gauge and finds it has 10 cm of water in it.
- While checking it, he accidentally knocks the rain gauge over and spills most of the water, leaving only 3 cm of water in the rain gauge.
- When he checks for the last time at 5 p.m., there is no change.

Graph A



Graph B

a. Which of the two graphs could represent Andre's story? Explain your reasoning.

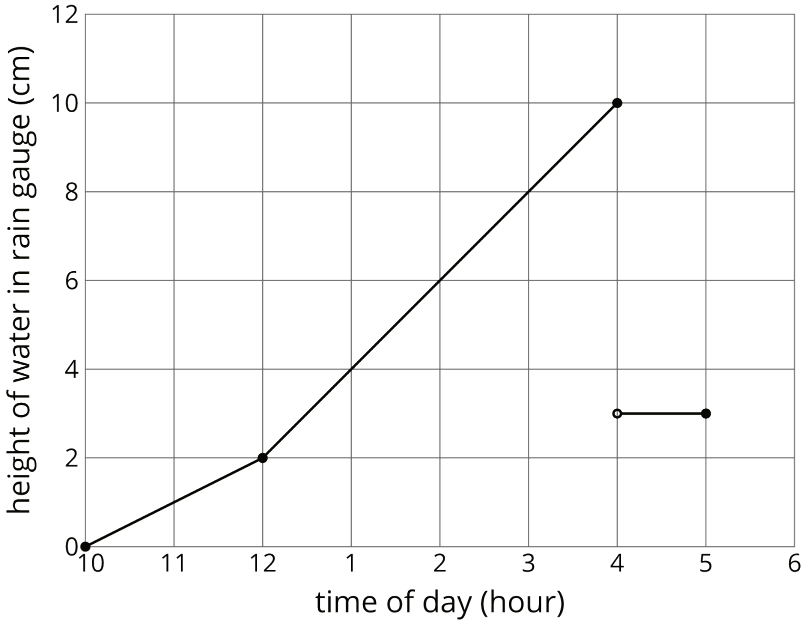
b. Label the axes of the correct graph with appropriate units.

c. Use the graph to determine how much total rain fell on Tuesday.

Solution

a. Graph A

b.



c. 10 cm of rain fell in total on Tuesday. No rain fell after Andre spilled the rain gauge.

(From Unit 5, Lesson 6.)