Lesson 11: Using an Algorithm to Divide Fractions

Goals

- Coordinate (orally) different strategies for dividing by a fraction.
- Find the quotient of two fractions, and explain (orally, in writing, and using other representations) the solution method.
- Generalize a process for dividing a number by a fraction, and justify (orally) why this can be abstracted as $n \cdot \frac{b}{a}$.

Learning Targets

- I can describe and apply a rule to divide numbers by any fraction.

Lesson Narrative

In the previous lesson, students began to develop a general algorithm for dividing a fraction by a fraction. They complete that process in this lesson. Students calculate quotients using the steps they observed previously (i.e., to divide by $\frac{a}{b}$, we can multiply by b and divide by a), and compare them to quotients found by reasoning with a tape diagram. Through repeated reasoning, they notice that the two methods produce the same quotient and that the steps can be summed up as an algorithm: to divide by $\frac{a}{b}$, we multiply by $\frac{b}{a}$ (MP8). As students use the algorithm to divide different numbers (whole numbers and fractions), they begin to see its flexibility and efficiency.

Alignments

Building On

- 5.NF.B.4: Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR7: Compare and Connect

- MLR8: Discussion Supports
- Think Pair Share

Student Learning Goals

Let's divide fractions using the rule we learned.

11.1 Multiplying Fractions

Warm Up: 5 minutes

This warm-up revisits multiplication of fractions from grade 5. Students will use this skill as they divide fractions throughout the lesson and the rest of the unit.

Building On

- 5.NF.B.4

Launch

Give students 2–3 minutes of quiet work time to complete the questions. Ask them to be prepared to explain their reasoning.

Student Task Statement

Evaluate each expression.

1. $\frac{2}{3} \cdot 27$
2. $\frac{1}{2} \cdot \frac{2}{3}$
3. $\frac{2}{9} \cdot \frac{3}{5}$
4. $\frac{27}{100} \cdot \frac{200}{9}$
5. $\left(1\frac{3}{4}\right) \cdot \frac{5}{7}$

Student Response

1. 18
2. $\frac{1}{3}$
3. $\frac{2}{15}$
4. 6
5. $\frac{5}{4}$ (or $1\frac{1}{4}$)

Activity Synthesis

Ask a student to share their answer and reasoning to each question, then ask if anyone disagrees. Invite students who disagree to share their explanations. If not mentioned in students' explanations, discuss strategies for multiplying fractions efficiently, how to multiply fractions in which a numerator and a denominator share at least one common factor, and how to multiply mixed numbers.

If students mention "canceling" a numerator and a denominator that share a common factor, demonstrate using the term "dividing" instead. For example, if a student suggests that in the second question ($\frac{1}{2} \cdot \frac{2}{3}$) the 2 in $\frac{1}{2}$ and the 2 in the $\frac{2}{3}$ "cancel out", rephrase the statement by saying that dividing the 2 in the numerator by the 2 in the denominator gives us 1, and multiplying by 1 does not change the other numerator or denominator.

11.2 Dividing a Fraction by a Fraction

15 minutes (there is a digital version of this activity)

This is the final task in a series that leads students toward a general procedure for dividing fractions. Students verify previous observations about the steps for dividing non-unit fractions (namely, multiplying by the denominator and dividing by the numerator) and contrast the results with those found using diagrams. They then generalize these steps as an algorithm and apply it to answer other division questions.

As students discuss in their groups, listen to their observations and explanations. Select students with clear explanations to share later.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR8: Discussion Supports
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 5–7 minutes of quiet think time and 2–3 minutes to share their response with their partner. Provide access to colored pencils. Some students may find it helpful to identify whole groups and partial groups on a tape diagram by coloring.

Students using the digital materials can use an applet to investigate division of fractions.

Access for Students with Disabilities

Representation: Internalize Comprehension. Provide additional examples and counterexamples to illustrate concepts. Students may benefit from additional examples of division problems that they can "test" to reinforce understanding of their conclusion about how to divide a number by any fraction.
Supports accessibility for: Visual-spatial processing; Organization

Access for English Language Learners

Conversing, Representing: MLR8 Discussion Supports. Provide the following sentence frames for students to use when they share their response with their partner: "I know there are ___ $\frac{1}{8}$s in $\frac{3}{4}$ because....", "I drew the diagram like this ___ because....", and "First, I ___ because....". This will help students produce statements that describe how to divide a number by any fraction.
Design Principle(s): Cultivate conversation; Support sense-making

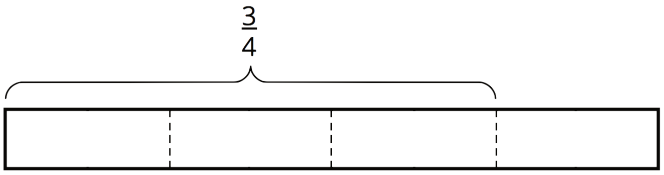
Student Task Statement

Work with a partner. One person works on the questions labeled "Partner A" and the other person works on those labeled "Partner B."

1. Partner A: Find the value of each expression by completing the diagram.

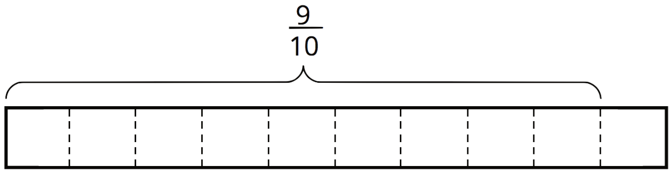
    a. $\frac{3}{4} \div \frac{1}{8}$

    How many $\frac{1}{8}$s in $\frac{3}{4}$?

    b. $\frac{9}{10} \div \frac{3}{5}$

    How many $\frac{3}{5}$s in $\frac{9}{10}$?

    Partner B:

    Elena said, "If I want to divide 4 by $\frac{2}{5}$, I can multiply 4 by 5 and then divide it by 2 or multiply it by $\frac{1}{2}$."

Find the value of each expression using the strategy Elena described.

a. $\frac{3}{4} \div \frac{1}{8}$

b. $\frac{9}{10} \div \frac{3}{5}$

2. What do you notice about the diagrams and expressions? Discuss with your partner.

3. Complete this sentence based on what you noticed:

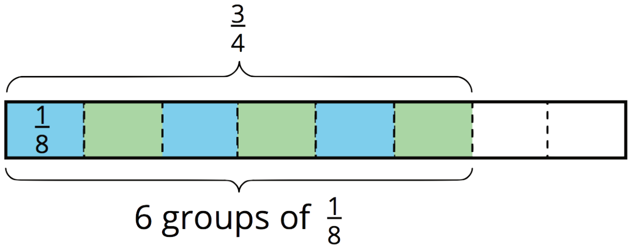
To divide a number n by a fraction $\frac{a}{b}$, we can multiply n by _______ and then divide the product by _______.

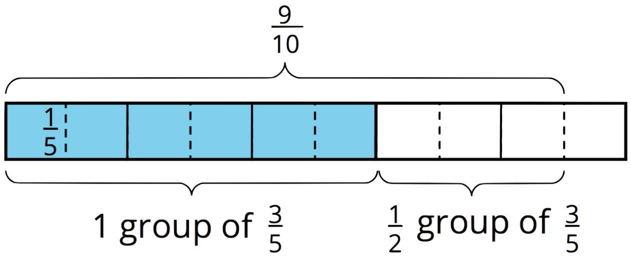
4. Select **all** the equations that represent the sentence you completed.

- $n \div \frac{a}{b} = n \cdot b \div a$
- $n \div \frac{a}{b} = n \cdot a \div b$
- $n \div \frac{a}{b} = n \cdot \frac{a}{b}$
- $n \div \frac{a}{b} = n \cdot \frac{b}{a}$

Student Response

1. Partner A

a. 

b. 

Partner B

a. $\frac{3}{4} \div \frac{1}{8} = \frac{3}{4} \cdot 8 = 6$

b. $\frac{9}{10} \div \frac{3}{5} = \frac{9}{10} \cdot 5 \div 3 = \frac{3}{2}$ or $1\frac{1}{2}$

2. Answers vary. Sample response: Multiplying by the denominator finds the total number of pieces in the diagram. Dividing by the numerator finds the number of groups. Sometimes you end up with a fraction of a group.
3. Multiply n by b and then divide the product by a.
4. $n \div \frac{a}{b} = n \cdot b \div a$ and $n \div \frac{a}{b} = n \cdot \frac{b}{a}$

Activity Synthesis

Invite a couple of students to share their conclusion about how to divide a number by any fraction. Then, review the sequence of reasoning that led us to this conclusion using both numerical examples and algebraic statements throughout. Remind students that in the past few activities, we learned that:

- Dividing by a whole number n is the same as multiplying by a unit fraction $\frac{1}{n}$ (e.g., dividing by 5 is the same as multiplying by $\frac{1}{5}$).
- Dividing by a unit fraction $\frac{1}{n}$ is the same as multiplying by a whole number n (e.g., dividing by $\frac{1}{7}$ is the same as multiplying by 7).
- Dividing by a fraction $\frac{a}{b}$ is the same as multiplying by a unit fraction $\frac{1}{a}$ and multiplying by a whole number b, which is the same as multiplying by $\frac{b}{a}$ (e.g., dividing by $\frac{5}{7}$ is the same as multiplying by $\frac{1}{5}$, and then by 7. Performing these two steps gives the same result as multiplying by $\frac{7}{5}$).

Finish the discussion by trying out the generalized method with other fractions such a $18 \div \frac{9}{7}$, or $\frac{15}{14} \div \frac{5}{2}$. Explain that although we now have a reliable and efficient method to divide any number by any fraction, sometimes it is still easier and more natural to think of the quotient in terms of a multiplication problem with a missing factor and to use diagrams to find the missing factor.

11.3 Using an Algorithm to Divide Fractions

15 minutes

This activity allows students to practice using the algorithm from earlier to solve division problems that involve a wider variety of fractions. Students can use any method of reasoning and are not expected to use the algorithm. As they encounter problems with less-friendly numbers, however, they notice that it becomes more challenging to use diagrams or other concrete strategies, and more efficient to use the algorithm. As they work through the activity, students choose their method.

Monitor the strategies students use and identify those with different strategies— including those who may not have used the algorithm—so they can share later.

Addressing

- 6.NS.A.1

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR7: Compare and Connect
- Think Pair Share

Launch

Keep students in groups of 2. Give students 5–7 minutes of quiet work time, followed by 2–3 minutes to discuss their responses with a partner.

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge. Provide students with access to blank tape diagrams. Encourage students to attempt more than one strategy for at least one of the problems.
Supports accessibility for: Visual-spatial processing; Organization

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Student Task Statement

Calculate each quotient. Show your thinking and be prepared to explain your reasoning.

1. $\frac{8}{9} \div 4$
2. $\frac{3}{4} \div \frac{1}{2}$
3. $3\frac{1}{3} \div \frac{2}{9}$
4. $\frac{9}{2} \div \frac{3}{8}$
5. $6\frac{2}{5} \div 3$
6. After biking $5\frac{1}{2}$ miles, Jada has traveled $\frac{2}{3}$ of the length of her trip. How long (in miles) is the entire length of her trip? Write an equation to represent the situation, and then find the answer.

Student Response

1. $\frac{8}{9} \cdot \frac{1}{4} = \frac{2}{9}$
2. $\frac{3}{4} \cdot 2 = \frac{3}{2}$ or $1\frac{1}{2}$

3. $\frac{10}{3} \cdot 9 \div 2 = 15$

4. $\frac{9}{2} \cdot \frac{8}{3} = 12$

5. $\frac{32}{5} \cdot \frac{1}{3} = \frac{32}{15}$ or $2\frac{2}{15}$

6. Equation: $\frac{2}{3} \cdot ? = 5\frac{1}{2}$ (or $? \cdot \frac{2}{3} = 5\frac{1}{2}$, $5\frac{1}{2} \div \frac{2}{3} = ?$, $5\frac{1}{2} \div ? = \frac{2}{3}$). Answer: $8\frac{1}{4}$ miles. Sample reasoning: $5\frac{1}{2} \div \frac{2}{3} = \frac{11}{2} \cdot \frac{3}{2} = \frac{33}{4} = 8\frac{1}{4}$

Are You Ready for More?

Suppose you have a pint of grape juice and a pint of milk. You pour 1 tablespoon of the grape juice into the milk and mix it up. Then you pour 1 tablespoon of this mixture back into the grape juice. Which liquid is more contaminated?

Student Response

1 tablespoon is $\frac{1}{32}$ of a pint. This means that the pint of milk with the tablespoon of grape juice is $1\frac{1}{32}$ pints of mixed liquid and is $\frac{32}{33}$ of milk. When a tablespoon of the mixture is added back into the grape juice, there's less than a tablespoon of milk being added. This means that the pint of milk and tablespoon of grape juice is more contaminated by grape juice.

Activity Synthesis

Select previously identified students to share their responses. Sequence their presentations so that students with the more concrete strategies (e.g., drawing pictures) share before those with more abstract strategies. Students using the algorithm should share last. Find opportunities to connect the different methods. For example, point out where the multiplication by a denominator and division by a numerator are visible in a tape diagram.

Access for English Language Learners

Conversing, Representing: MLR7 Compare and Connect. As students consider the different strategies, invite them to make connections between the various representations and approaches. Ask, "What do each of the strategies have in common?", "How are the strategies different?" and "Which strategy is more efficient? Why?" Listen for and amplify observations that include mathematical language and reasoning.
Design Principle(s): Maximize meta-awareness; Optimize output (for comparison)

Lesson Synthesis

In this lesson, we noticed a more-efficient way to divide fractions. We found that to divide $\frac{3}{2}$ by $\frac{2}{5}$, for example, we can multiply $\frac{3}{2}$ by 5 and then by $\frac{1}{2}$, or simply multiply $\frac{3}{2}$ by $\frac{5}{2}$.

Let's see how this is the same or different than finding the quotient using tape diagrams. (If time permits, consider illustrating each diagram for all to see.)

- “Suppose we interpret $\frac{3}{2} \div \frac{2}{5}$ to mean ‘how many $\frac{2}{5}$ are in $\frac{3}{2}$?’ and use a tape diagram to find the answer. Where do we see the multiplication by 5 and by $\frac{1}{2}$ in the diagramming process?” (We draw a diagram to represent $\frac{3}{2}$ and draw equal parts, each with a value of $\frac{1}{5}$. We count how many groups of $\frac{2}{5}$ there are. Partitioning into fifths gives as 5 times as many parts. This is the multiplication by 5. Counting by two-fifths leads to half as many parts. This is the multiplication by $\frac{1}{2}$.)

- “Suppose we interpret $\frac{3}{2} \div \frac{2}{5}$ to mean ‘$\frac{2}{5}$ of what number is $\frac{3}{2}$?’ and use a tape diagram to find the answer. Where do we see the multiplication by 5 and by $\frac{1}{2}$ in the diagramming process?” (We draw a tape diagram to represent a whole group. We mark two-fifths of it as having a value of $\frac{3}{2}$. We divide that value by 2 (or multiply by $\frac{1}{2}$) to find one fifth of a group. To find out how much is in the whole group, we multiply by 5.)

Note that in both cases, there is a multiplication by $\frac{1}{2}$ and another multiplication by 5, which is the same as multiplication by $\frac{5}{2}$. Highlight that dividing by $\frac{a}{b}$ is equivalent to multiplying by b and then by $\frac{1}{a}$, or simply multiplying by $\frac{b}{a}$ (the reciprocal of $\frac{a}{b}$). This is true whether we interpreted the division problem in terms of finding the number of groups or finding the size of a group.

11.4 Watering A Fraction of House Plants

Cool Down: 5 minutes

Addressing

- 6.NS.A.1

Student Task Statement

1. Find the value of $\frac{24}{25} \div \frac{4}{5}$. Show your reasoning.

2. If $\frac{4}{3}$ liters of water are enough to water $\frac{2}{5}$ of the plants in the house, how much water is necessary to water all the plants in the house? Write an equation to represent the situation, and then find the answer.

Student Response

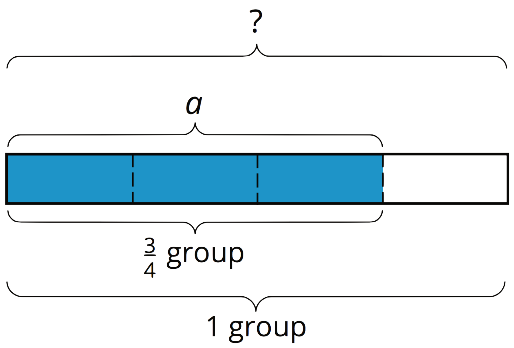
1. $\frac{6}{5}$, because $\frac{24}{25} \boldsymbol{\cdot} \frac{5}{4} = \frac{6}{5}$.

2. Equations: $\frac{4}{3} \div \frac{2}{5} = ?$ or $\frac{4}{3} \div ? = \frac{2}{5}$ (or $\frac{2}{5} \boldsymbol{\cdot} ? = \frac{4}{3}$ or $? \boldsymbol{\cdot} \frac{2}{5} = \frac{4}{3}$).

   Solution: $\frac{4}{3} \boldsymbol{\cdot} \frac{5}{2} = \frac{10}{3}$ (or $3\frac{1}{3}$)

Student Lesson Summary

The division $a \div \frac{3}{4} = ?$ is equivalent to $\frac{3}{4} \cdot ? = a$, so we can think of it as meaning "$\frac{3}{4}$ of what number is a?" and represent it with a diagram as shown. The length of the entire diagram represents the unknown number.



If $\frac{3}{4}$ of a number is a, then to find the number, we can first divide a by 3 to find $\frac{1}{4}$ of the number. Then we multiply the result by 4 to find the number.

The steps above can be written as: $a \div 3 \cdot 4$. Dividing by 3 is the same as multiplying by $\frac{1}{3}$, so we can also write the steps as: $a \cdot \frac{1}{3} \cdot 4$.

In other words: $a \div 3 \cdot 4 = a \cdot \frac{1}{3} \cdot 4$. And $a \cdot \frac{1}{3} \cdot 4 = a \cdot \frac{4}{3}$, so we can say that:

$$a \div \frac{3}{4} = a \cdot \frac{4}{3}$$

In general, dividing a number by a fraction $\frac{c}{d}$ is the same as multiplying the number by $\frac{d}{c}$, which is the reciprocal of the fraction.

Lesson 11 Practice Problems

Problem 1

Statement

Select **all** the statements that show correct reasoning for finding $\frac{14}{15} \div \frac{7}{5}$.

A. Multiplying $\frac{14}{15}$ by 5 and then by $\frac{1}{7}$.

B. Dividing $\frac{14}{15}$ by 5, and then multiplying by $\frac{1}{7}$.

C. Multiplying $\frac{14}{15}$ by 7, and then multiplying by $\frac{1}{5}$.

D. Multiplying $\frac{14}{15}$ by 5 and then dividing by 7.

E. Multiplying $\frac{15}{14}$ by 7 and then dividing by 5.

Solution

["A", "D"]

Problem 2

Statement

Clare said that $\frac{4}{3} \div \frac{5}{2}$ is $\frac{10}{3}$. She reasoned: $\frac{4}{3} \cdot 5 = \frac{20}{3}$ and $\frac{20}{3} \div 2 = \frac{10}{3}$.

Explain why Clare’s answer and reasoning are incorrect. Find the correct quotient.

Solution

The correct quotient is $\frac{8}{15}$. Explanations vary. Sample response:

- Clare should have multiplied $\frac{4}{3}$ by 2 to find how many groups of $\frac{1}{2}$ are in $\frac{4}{3}$ and then divide the result by 5.
- Clare divided the fraction $\frac{4}{3}$ by the fraction $\frac{2}{5}$ instead of $\frac{5}{2}$.

Problem 3

Statement

Find the value of $\frac{15}{4} \div \frac{5}{8}$. Show your reasoning.

Solution

6. Reasoning varies. Sample reasoning: There are $\frac{15}{4} \cdot 8$ or 30 groups of $\frac{1}{8}$ in $\frac{15}{4}$. If five $\frac{1}{8}$s make a group, then the number of groups is $\frac{1}{5}$ of 30, which is 6.

Problem 4

Statement

Consider the problem: Kiran has $2\frac{3}{4}$ pounds of flour. When he divides the flour into equal-sized bags, he fills $4\frac{1}{8}$ bags. How many pounds fit in each bag?

Write a multiplication equation and a division equation to represent the question. Then, find the answer and show your reasoning.

Solution

$\frac{2}{3}$ pound per bag. Reasoning varies. Sample reasoning: $4\frac{1}{8} \cdot ? = 2\frac{3}{4}$ can be written as $2\frac{3}{4} \div 4\frac{1}{8} = ?$. Using the algorithm to divide: $2\frac{3}{4} \div 4\frac{1}{8} = \frac{11}{4} \div \frac{33}{8} = \frac{11}{4} \cdot \frac{8}{33} = \frac{2}{3}$.

Problem 5

Statement

Divide $4\frac{1}{2}$ by each of these unit fractions.

a. $\frac{1}{8}$

b. $\frac{1}{4}$

c. $\frac{1}{6}$

Solution

a. 36

b. 18

c. 27

(From Unit 4, Lesson 10.)

Problem 6

Statement

Consider the problem: After charging for $\frac{1}{3}$ of an hour, a phone is at $\frac{2}{5}$ of its full power. How long will it take the phone to charge completely?

Decide whether each equation can represent the situation.

a. $\frac{1}{3} \cdot ? = \frac{2}{5}$

b. $\frac{1}{3} \div \frac{2}{5} = ?$

c. $\frac{2}{5} \div \frac{1}{3} = ?$

d. $\frac{2}{5} \cdot ? = \frac{1}{3}$

Solution

a. No

b. Yes

c. No

d. Yes

(From Unit 4, Lesson 9.)

Problem 7

Statement

Elena and Noah are each filling a bucket with water. Noah's bucket is $\frac{2}{5}$ full and the water weighs $2\frac{1}{2}$ pounds. How much does Elena's water weigh if her bucket is full and her bucket is identical to Noah's?

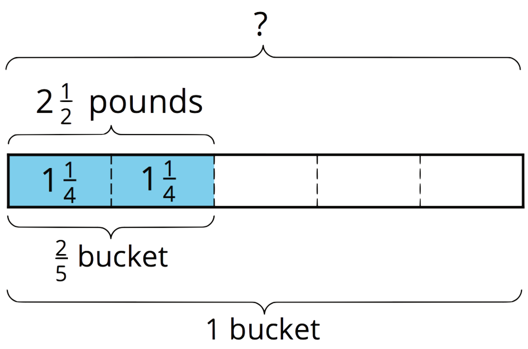
a. Write multiplication and division equations to represent the question.

b. Draw a diagram to show the relationship between the quantities and to find the answer.

Solution

a. $\frac{2}{5} \cdot ? = 2\frac{1}{2}$ (or equivalent), $2\frac{1}{2} \div \frac{2}{5} = ?$

b. $6\frac{1}{4}$ pounds. Sample diagram:



(From Unit 4, Lesson 8.)