Lesson 7: All, Some, or No Solutions

Goals

- Compare and contrast (orally and in writing) equations that have no solutions or infinitely many solutions.
- Create linear equations in one variable that have either no solutions or infinitely many solutions, using structure, and explain (orally) the solution method.

Learning Targets

- I can determine whether an equation has no solutions, one solution, or infinitely many solutions.

Lesson Narrative

In previous lessons, students have mostly worked with equations that have exactly one solution and have solved those equations by a sequence of steps that lead to an equation of the form $x = \text{number}$. In this lesson they encounter equations that have no solutions and equations for which every number is a solution. In the first case, when students try to solve the equation, they end up with false statement like $0 = 5$. In the second case, they end up with a statement that is always true, such as $6x = 6x$. In preparation for the next lesson, where students will learn to predict the number of solutions from the structure of an equation, students complete equations in three different ways to make them have no solution, one solution, or infinitely many solutions.

Alignments

Addressing

- 8.EE.C.7.a: Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers).

Building Towards

- 8.EE.C.7.a: Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers).

Instructional Routines

- MLR2: Collect and Display
- MLR8: Discussion Supports
- Which One Doesn’t Belong?

Student Learning Goals

Let's think about how many solutions an equation can have.

7.1 Which One Doesn't Belong: Equations

Warm Up: 5 minutes

The purpose of this warm-up is for students to think about equality and properties of operations when deciding whether equations are true. While there are many reasons students may decide one equation doesn't belong, highlight responses that mention both sides of the equation being equal and ask students to explain how they can tell.

Building Towards

- 8.EE.C.7.a

Instructional Routines

- Which One Doesn't Belong?

Launch

Arrange students in groups of 2–4. Give students 1 minute of quiet think time. Ask students to indicate when they have noticed one equation that does not belong and can explain why not. Give students time to share their thinking with their group. After everyone has conferred in groups, ask the group to offer at least one reason *each* equation doesn't belong.

Student Task Statement

Which one doesn't belong?

1. $5 + 7 = 7 + 5$
2. $5 \cdot 7 = 7 \cdot 5$
3. $2 = 7 - 5$
4. $5 - 7 = 7 - 5$

Student Response

Answers vary. Possible solutions: 1 is different because it is the only one that involves addition explicitly. 2 is different because it is the only one that involves multiplication. 3 is different because it has a 2 in it and all the others only include 5 and 7. 3 is different because it has a single number on one side and all the others have two numbers on both sides. 4 is different because it is not true.

Activity Synthesis

After students have conferred in groups, invite each group to share one reason why a particular equation might not belong. Record and display the responses for all to see. After each response, ask the rest of the class if they agree or disagree. Since there is no single correct answer to the question of which equation does not belong, attend to students' explanations and ensure the reasons given are correct.

If no students point out that 4 is not true, ask if all of the equations are true and to explain how they know.

7.2 Thinking About Solutions

15 minutes
Students who pause to think about the structure of a complex equation before taking steps to solve it can find the most efficient solution paths and, sometimes, notice that there is no single solution to be found. The goal of this lesson is to encourage students to make this pause part of their routine and to build their skill at understanding and manipulating the structure of equations through the study of two special types of equations: ones that are always true and ones that are never true.

Students begin the activity sorting a variety of equations into categories based on their number of solutions. The activity ends with students filling in the blank side of an equation to make an equation that is always true and then again to make an equation that is never true.

Addressing

- 8.EE.C.7.a

Instructional Routines

- MLR2: Collect and Display

Launch

Display the equation $2t + 5 = 2t + 5$ and ask students to find a value of t that makes the equation true. After a brief quiet think time, record the responses of a few students next to the equation. Ask the class if they think there is any value of t that doesn't work and invite students to explain why or why not. If no students suggest seeing what happens if you try to solve for t, demonstrate that no matter what steps you take, the equation will always end with a statement that is always true such as $t = t$ or $5 = 5$.

Next, display the equation $n + 5 = n + 7$ and ask students to find a value of n that makes the equation true. After a brief quiet think time, ask the class if they think there might be a value that works and select a few students to explain why or why not. While you can try and solve for n here as with the previous example, encourage students to also use the logic that adding different values to the same value cannot result in two numbers that are the same.

Tell students that these are two special kinds of equations. The first equation has many solutions—it is true for all values of t. Remind students that they encountered this type of equation during the number trick activity where one side of the equation looked complicated but it was actually the same as a very simple expression, which is why the trick worked. The second equation has no solutions—it is not true for any values of n.

Arrange students in groups of 2. Give students 3–5 minutes quiet work time for the first problem, followed by partner discussion to share how they sorted the equations. Give time for partners to complete the remaining problems and follow with a whole-class discussion.

Access for English Language Learners

Representing: MLR2 Collect and Display. As groups of students discuss how they sorted the equations, circulate and record the language students use to justify their decisions on a visual display. Ask students to describe the reasons for their selection, and to name what these equations have in common. Listen for phrases such as “variables with the same coefficient” or “the variable was eliminated.” Consider dividing the display into sections labeled “true for all values” and “true for no values,” and group words and phrases in the appropriate area. Remind students to borrow language from the display as needed. This will help students use mathematical language to describe their reasoning and increase awareness about what these types of equations look like.
Design Principle(s): Support sense-making

Anticipated Misconceptions

For the last part of the activity, students may think any expression that is not equivalent to $6u - 10$ is a good answer. Remind students that there is another possibility: that the equation will have one solution. For example, the expression $3u + 5$ does allow for a solution.

Student Task Statement

$$n = n$$

$$2t + 6 = 2(t + 3)$$

$$3(n + 1) = 3n + 1$$

$$\frac{1}{4}(20d + 4) = 5d$$

$$5 - 9 + 3x = \text{-}10 + 6 + 3x$$

$$\frac{1}{2} + x = \frac{1}{3} + x$$

$$y \cdot \text{-}6 \cdot \text{-}3 = 2 \cdot y \cdot 9$$

$$v + 2 = v - 2$$

1. Sort these equations into the two types: true for all values and true for no values.
2. Write the other side of this equation so that this equation is true for all values of u.
$$6(u - 2) + 2 =$$
3. Write the other side of this equation so that this equation is true for no values of u.
$$6(u - 2) + 2 =$$

Student Response

1. True for all values: $n = n$, $y \cdot \text{-}6 \cdot \text{-}3 = 2 \cdot y \cdot 9$, $2t + 6 = 2(t + 3)$, $5 - 9 + 3x = \text{-}10 + 6 + 3x$
   True for no values: $\frac{1}{2} + x = \frac{1}{3} + x$, $3(n + 1) = 3n + 1$, $\frac{1}{4}(20d + 4) = 5d$, $v + 2 = v - 2$
2. Answers vary. Sample response: $6(u - 2) + 2 = 6u - 10$

3. Answers vary. Sample response: $6(u-2)+2=6u$

Are You Ready for More?

Consecutive numbers follow one right after the other. An example of three consecutive numbers is 17, 18, and 19. Another example is -100, -99, -98.

How many sets of two or more consecutive positive integers can be added to obtain a sum of 100?

Student Response

There are two sets. The first has 8 consecutive integers: 9, 10, 11, 12, 13, 14, 15, 16. The second has five consecutive integers: 18, 19, 20, 21, 22.

Activity Synthesis

Display a list of the equations from the task with space to add student ideas next to the equations. The purpose of this discussion is for students to see multiple ways of thinking about and justifying the number of solutions an equation has.

Invite students to choose an equation, say if it is true for all values or true for no values, and then justify how they know. Continue until the solutions to all the equations are known. Record a summarized version of the student's solution next to the equation.

Next, ask students for different ways to write the other side of the equation for the second problem and add these to the display. For example, students may have distributed $6(u-2)+2$ to get $6u-12+2$ while others chose $6u-10$ or something with more terms, such as $6(u-2+1)-4$.

End the discussion by asking students for different ways to write the other side of the incomplete equation in the last question. It is important to note, if no students point it out, that all solutions should be equivalent to $6u+$ _ where the blank represents any number other than -10.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Provide students with a graphic organizer to support their participation during the synthesis. Invite students to describe what to look for to determine whether an equation is true for all values or true for no values, and to include examples for each.
Supports accessibility for: Conceptual processing; Organization

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7.3 What's the Equation?

15 minutes
In this activity, students are presented with three equations all with a missing term. They are asked to fill in the missing term to create equations with either no solution or infinitely many solutions,

building on the work begun in the previous activity. At the end, students summarize what they have learned about how to tell if an equation is true for all values of x or no values of x.

Addressing

- 8.EE.C.7.a

Instructional Routines

- MLR8: Discussion Supports

Launch

Give students 3–5 minutes of quiet think time followed by 3–5 minutes of partner discussion. Follow with a whole-class discussion.

Access for Students with Disabilities

Action and Expression: Develop Expression and Communication. To help get students started during partner discussion, display sentence frames such as "Equations that are always true for x have _____", "Equations which have no solution for any value of x have _____" .
Supports accessibility for: Language; Organization

Student Task Statement

1. Complete each equation so that it is true for all values of x.
    a. $3x + 6 = 3(x + \_\_)$
    b. $x - 2 = -(\_\_ - x)$
    c. $\frac{15x-10}{5} = \_\_ - 2$
2. Complete each equation so that it is true for no values of x.
    a. $3x + 6 = 3(x + \_\_)$
    b. $x - 2 = -(\_\_ - x)$
    c. $\frac{15x-10}{5} = \_\_ - 2$
3. Describe how you know whether an equation will be true for all values of x or true for no values of x.

Student Response

1. a. 2
    b. 2
    c. $3x$
2. a. Answers vary. Any number other than 2 will give an equation with no solution.

b. Answers vary. Any number other than 2 will give an equation with no solution.

c. Answers vary. Any expression of the form ($3x$+a number other than 0) will give an equation with no solution. Note: A numerical answer will yield a linear equation of one variable which has one solution.

3. Explanations vary. Sample response: Equations which are always true for any value of x have equivalent expressions on each side. Equations which which have no solution for any value of x simplify to a statement of two unequal numbers being equal, which is always false.

Activity Synthesis

Display each equation with a large space for writing. Under each equation, invite students to share what they used to make the equation be true for all values of x and record these for all to see. Ask:

- “What did all these answers have in common?” (There is only one possible answer for each equation that will make it be always true.)
- “What strategy did you use to figure out what that answer had to be?” (The solution had to be something that would make the right side equivalent to the left.)

Next, invite students to share what they used to make the equation true for no values of x and record these for all to see. Ask:

- “Why are there so many different solutions for these questions?” (As long as the answer wasn't what we chose in part 1, then the equation will never have a solution.)
- “What was different about Equation C?” (We had to be careful to make sure that the variable coefficient was 3 and we added a constant so that the equation wouldn't have a single solution.)

Ask students to share observations they made for the last question. If no student points it out, explain that an equation with no solution can always be rearranged or manipulated to say that two unequal values are equal (e.g., 2=3), which means the equation is never true.

Access for English Language Learners

Speaking: MLR8 Discussion Supports. Use this routine to support whole-class discussion. For each observation that is shared for the last question, ask students to restate and/or revoice what they heard using mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. This will provide more students with an opportunity to describe what they have learned about how to tell if an equation is true for all values of x or no values of x.
Design Principle(s): Support sense-making

Lesson Synthesis

Ask students to think about some ways they were able to determine how many solutions there were to the equations they solved today. Invite students to share some thing they did. For example, students may suggest:

- tested different values for the variable
- applied allowable moves to generate equivalent equations
- examined the structure of the equation

Ask students to write a short letter to someone taking the class next year about what they should look for when trying to decide how many solutions an equation has. Tell students to use examples, share any struggles they had in deciding on the number of solutions, and which strategies they prefer for figuring out the number of solutions.

7.4 Choose Your Own Solution

Cool Down: 5 minutes

Addressing

- 8.EE.C.7.a

Student Task Statement

$$3x + 8 = 3x +$$

What value could you write in after $3x$ that would make the equation true for:

1. no values of x?
2. all values of x?
3. just one value of x?

Student Response

1. Answers vary. Sample response: 7. The equation $3x + 8 = 3x + 7$ has no solutions. If you triple a number and add 8 to it, and triple the same number and add 7 to it, the results will never be equal, no matter what number you choose.
2. 8. The equation $3x + 8 = 3x + 8$ has many solutions. If you triple a number and add 8 to it, and triple the same number and add 8 to it, the results will always be equal, no matter what number you choose.
3. Answers vary. Sample response: x. The equation $3x + 8 = 3x + x$ has one solution. Students should add some variable term in order to create an equation with one solution.

Student Lesson Summary

An equation is a statement that two expressions have an equal value. The equation

$$2x = 6$$

is a true statement if x is 3:

$$2 \cdot 3 = 6$$

It is a false statement if x is 4:

$$2 \cdot 4 = 6$$

The equation $2x = 6$ has *one and only one solution*, because there is only one number (3) that you can double to get 6.

Some equations are true no matter what the value of the variable is. For example:

$$2x = x + x$$

is always true, because if you double a number, that will always be the same as adding the number to itself. Equations like $2x = x + x$ have an *infinite number of solutions*. We say it is true for all values of x.

Some equations have *no solutions*. For example:

$$x = x + 1$$

has no solutions, because no matter what the value of x is, it can't equal one more than itself.

When we solve an equation, we are looking for the values of the variable that make the equation true. When we try to solve the equation, we make allowable moves assuming it *has* a solution. Sometimes we make allowable moves and get an equation like this:

$$8 = 7$$

This statement is false, so it must be that the original equation had no solution at all.

Lesson 7 Practice Problems

Problem 1

Statement

For each equation, decide if it is always true or never true.

a. $x - 13 = x + 1$

b. $x + \frac{1}{2} = x - \frac{1}{2}$

c. $2(x+3) = 5x + 6 - 3x$

d. $x - 3 = 2x - 3 - x$

e. $3(x-5) = 2(x-5) + x$

Solution

a. Never true

b. Never true

c. Always true

d. Always true

e. Never true

Problem 2

Statement

Mai says that the equation $2x + 2 = x + 1$ has no solution because the left hand side is double the right hand side. Do you agree with Mai? Explain your reasoning.

Solution

Answers vary. Sample response: Mai is correct that $2x + 2 = 2(x+1)$, so the left hand side in this equation is double the right hand side. But -x and -1 can be added to both sides of the equation to get $x + 1 = 0$. So $x =$ -1 is a solution. (This works because 0 is its own double, and it is the only number that is its own double.)

Problem 3

Statement

a. Write the other side of this equation so it's true for all values of x: $\frac{1}{2}(6x - 10) - x =$

b. Write the other side of this equation so it's true for no values of x: $\frac{1}{2}(6x - 10) - x =$

Solution

a. $2x - 5$ (or equivalent)

b. Answers vary. Sample response: $2x + 5$

Problem 4

Statement

Here is an equation that is true for all values of x: $5(x+2)=5x+10$. Elena saw this equation and says she can tell $20(x+2)+31=4(5x+10)+31$ is also true for any value of x. How can she tell? Explain your reasoning.

Solution

Responses vary. Sample response: One could distribute the left side of the equation and show it is equal to the right side, but it is easier to see that each side of the original equation has been multiplied by 4 and added to 31. These moves keep both sides of the equation in balance, and so whatever values of x make the first equation true also make the second equation true.

Problem 5

Statement

Elena and Lin are trying to solve $\frac{1}{2}x+3=\frac{7}{2}x+5$. Describe the change they each make to each side of the equation.

a. Elena's first step is to write $3=\frac{7}{2}x-\frac{1}{2}x+5$.

b. Lin's first step is to write $x+6=7x+10$.

Solution

a. Elena subtracted $\frac{1}{2}x$ from each side.

b. Lin multiplied each side by 2.

(From Unit 4, Lesson 4.)

Problem 6

Statement

Solve each equation and check your solution.

$3x-6=4(2-3x)-8x$ $\quad$ $\frac{1}{2}z+6=\frac{3}{2}(z+6)$ $\quad$ $9-7w=8w+8$

Solution

a. $x=\frac{14}{23}$

b. $z=$ -3

c. $w = \frac{1}{15}$

(From Unit 4, Lesson 6.)

Problem 7

Statement

The point $(-3, 6)$ is on a line with a slope of 4.

a. Find two more points on the line.

b. Write an equation for the line.

Solution

a. Answers vary. Sample response: $(-2, 10)$, $(-1, 14)$

b. $y = 4x + 18$ (or equivalent)

(From Unit 3, Lesson 12.)