Lesson 3: Revisiting Proportional Relationships

Goals

- Calculate and interpret (orally) the constant of proportionality for a proportional relationship involving fractional quantities.
- Explain (orally and in writing) how to use a table with only two rows to solve a problem involving a proportional relationship.
- Write an equation to represent a given proportional relationship with a fractional constant of proportionality.

Learning Targets

- I can use a table with 2 rows and 2 columns to find an unknown value in a proportional relationship.
- When there is a constant rate, I can identify the two quantities that are in a proportional relationship.

Lesson Narrative

In grade 6 students solved ratio problems by reasoning about scale factors or unit rates. In grade 7 they see the two quantities in a set of equivalent ratios as being in a proportional relationship and move towards using the constant of proportionality to find missing numbers. This is useful in the sorts of tasks they are studying in this unit because the tasks involve repeatedly applying the same number (for example, a unit price) to different amounts. The unit price is a constant of proportionality between the amount purchased and the amount paid. When students describe the proportional relationship behind the repeated operation of finding the amount paid, they are engaging in MP8.

In this lesson students move toward solving problems involving proportional relationships by more efficient methods, especially by setting up and reasoning about a two-row table of equivalent ratios. This method encourages them to use the constant of proportionality rather than equivalent ratios.

Alignments

Building On

- 6.RP.A.3: Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

Addressing

- 7.RP.A.1: Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks

$1/2$ mile in each $1/4$ hour, compute the unit rate as the complex fraction $\frac{1/2}{1/4}$ miles per hour, equivalently 2 miles per hour.

- 7.RP.A.2: Recognize and represent proportional relationships between quantities.

Building Towards

- 7.RP.A.1: Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks $1/2$ mile in each $1/4$ hour, compute the unit rate as the complex fraction $\frac{1/2}{1/4}$ miles per hour, equivalently 2 miles per hour.
- 7.RP.A.2: Recognize and represent proportional relationships between quantities.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR1: Stronger and Clearer Each Time
- MLR2: Collect and Display
- MLR3: Clarify, Critique, Correct
- Think Pair Share

Student Learning Goals

Let’s use constants of proportionality to solve more problems.

3.1 Recipe Ratios

Warm Up: 5 minutes

The purpose of this warm-up is to bring up two main methods for figuring out missing numbers in a table that represents a proportional relationship. The two methods students might use for this activity are:

- Using a scale factor to find equivalent ratios, e.g. multiplying the first row by $1\frac{1}{2}$ to get the second row.
- Using the constant of proportionality, 2, between the first column and the second column.

This activity is to get students thinking about the second method as a more efficient method, since it works for every row. This lays the groundwork for solving problems using proportional relationships and for the activities in this lesson (specifically, the activity right after the warm up).

Building On

- 6.RP.A.3

Building Towards

- 7.RP.A.1
- 7.RP.A.2

Instructional Routines

- Think Pair Share

Launch

In a previous unit, students worked extensively with sets of equivalent ratios that represented ingredients for different numbers of batches of a recipe. If necessary, remind them how this works. For example, you might say "This recipe calls for $\frac{1}{2}$ cup of sugar and 1 cup of flour. What if I wanted to make half a batch of the recipe? What if I wanted to make 5 batches?"

Arrange students in groups of 2. Give 1 minute of quiet work time followed by time to compare their table with a partner and a whole-class discussion.

Anticipated Misconceptions

Some students may assume the sugar column will continue to increase by the same amount without paying close attention to the values in the flour column. Ask these students what they notice about the values in the flour column and if it makes sense for the sugar amount to increase by the same amount each time.

Student Task Statement

A recipe calls for $\frac{1}{2}$ cup sugar and 1 cup flour. Complete the table to show how much sugar and flour to use in different numbers of batches of the recipe.

sugar (cups)	flour (cups)
$\frac{1}{2}$	1
$\frac{3}{4}$	
	$1\frac{3}{4}$
1	
	$2\frac{1}{2}$

Student Response

sugar (cups)	flour (cups)
$\frac{1}{2}$	1
$\frac{3}{4}$	$1\frac{1}{2}$, because the recipe uses twice as much flour as sugar, and $2 \cdot \frac{3}{4} = 1\frac{1}{2}$.
$\frac{7}{8}$, because the recipe uses half as much sugar as flour, and $\frac{1}{2} \cdot 1\frac{3}{4} = \frac{1}{2} \cdot \frac{7}{4} = \frac{7}{8}$.	$1\frac{3}{4}$
1	2, because the recipe uses twice as much flour as sugar, and $2 \cdot 1 = 2$.
$1\frac{1}{4}$, because the recipe uses half as much sugar as flour, and $\frac{1}{2} \cdot 2\frac{1}{2} = 1\frac{1}{4}$.	$2\frac{1}{2}$

Activity Synthesis

Display the table for all to see and ask students to share the answers they calculated for each missing entry. After the table is complete, ask the students if they agree or disagree with the values in the table. Select students to share who used the scale factor or constant of proportionality methods to find the equivalent ratios to share their reasoning. Record their ideas directly on the table if possible and display for all to see.

3.2 The Price of Rope

15 minutes

The purpose of this activity is to ensure students understand how an abbreviated table can be used to solve a problem (the most abbreviated table consists of only two rows). Students may have done some work like this in grade 6, in which case this activity serves to reinforce and remind. For teachers accustomed to procedures for “setting up a proportion,” this approach is very similar, except that students have the column headings to help make sure they get the numbers in the right places. Also, they should get a better idea for why they are multiplying and dividing, because they are finding and using either a scale factor or the constant of proportionality. (Note that using the constant of proportionality is easier, and also the natural way you would think about calculating the price of any amount of something.)

Building On

- 6.RP.A.3

Building Towards

- 7.RP.A.1
- 7.RP.A.2

Instructional Routines

- MLR2: Collect and Display
- Think Pair Share

Launch

Students should be comfortable with Kiran's method from their work with tables of equivalent ratios in grade 6. However, if needed, show them this even longer solution method first and let them examine it. Ask why Lin decided to multiply by $\frac{1}{3}$. Once students are comfortable with the reasoning shown, explain that you will be looking at more efficient ways of solving this problem with a table.

Lin's method:

	length of rope (feet)	price of rope (dollars)	
	6	7.50	
$\cdot 2$	12	15.00	$\cdot 2$
$\cdot 2$	24	30.00	$\cdot 2$
$\cdot 2$	48	60.00	$\cdot 2$
$\cdot \frac{1}{3}$	2	2.50	$\cdot \frac{1}{3}$
$\cdot 25$	50	62.50	$\cdot 25$

Arrange students in groups of 2. Give students 2–3 minutes of quiet work time and then time to discuss their solutions with their partner. Follow with a whole-class discussion around the method they think Priya used.

Access for English Language Learners

Speaking: MLR2 Collect and Display. While pairs are working, listen for and collect vocabulary and phrases students use to explain the similarities and differences between the scale factor method and the constant of proportionality method. On a display, organize the responses into two columns, one for each method. Throughout the remainder of the lesson, continue to update collected student language and remind students to borrow language from the display as needed. This will help students use mathematical language during paired and group discussions.
Design Principle(s): Optimize output (for explanation); Support sense-making

Anticipated Misconceptions

Some students may struggle to progress with Priya's method because the arrows are not drawn in the image and none of the values given are easily divisible. There are many supporting questions that could be asked.

- What if we knew the price of 1 foot of rope?
- If 6 times something is 7.5, how can we find the something?

Student Task Statement

Two students are solving the same problem: At a hardware store, they can cut a length of rope off of a big roll, so you can buy any length you like. The cost for 6 feet of rope is $7.50. How much would you pay for 50 feet of rope, at this rate?

1. Kiran knows he can solve the problem this way.

	length of rope (feet)	price of rope (dollars)	
	6	7.50	
$\cdot\frac{1}{6}$	1	1.25	$\cdot\frac{1}{6}$
$\cdot 50$	50		$\cdot 50$

   What would be Kiran's answer?

2. Kiran wants to know if there is a more efficient way of solving the problem. Priya says she can solve the problem with only 2 rows in the table.

length of rope (feet)	price of rope (dollars)
6	7.50
50	

What do you think Priya's method is?

Student Response

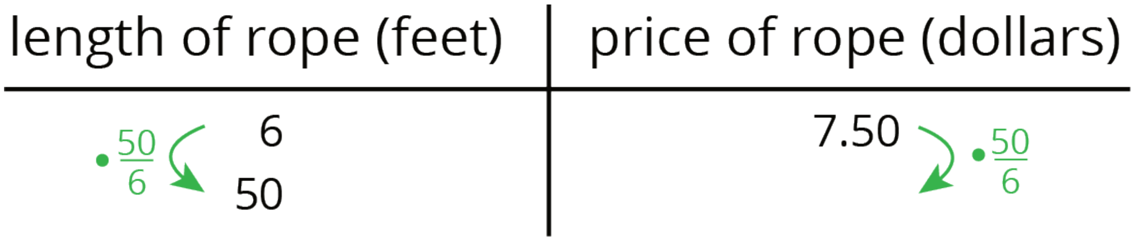
1. \$62.50, because $(1.25) \cdot 50 = 62.5$

2. Answers vary. Sample responses:
   - (preferred) 1 foot of rope costs \$1.25 because $7.5 \div 6 = 1.25$. (Or, $6 \cdot 1.25 = 7.5$.) So multiply 50 by 1.25 to find the cost of 50 feet of rope. $50 \cdot (1.25) = 62.5$.
   - Since $50 \div 6 = 8\frac{1}{3}$, that means $6 \cdot 8\frac{1}{3} = 50$. So multiply 7.5 by $8\frac{1}{3}$ to get 62.5.

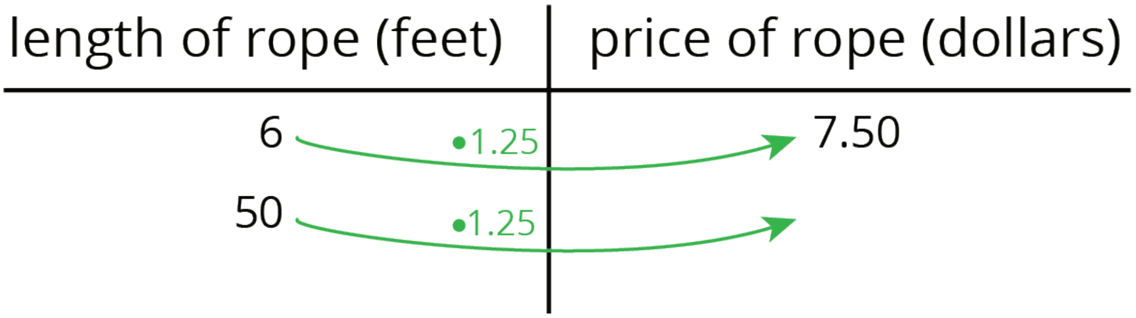
Activity Synthesis

Ask selected students to show the way they solved the problem. If no students come up with one of these methods, display it for all to see.

Scale factor method:



Constant of proportionality method:



Ask students:

- “How did you find the scale factor?”
- “How did you find the constant of proportionality?”
- “What does the constant of proportionality (1.25) mean in this context?”
- “Which method is your preference to use? Why?”

Although either method will work, there are reasons to prefer using the constant of proportionality to approach problems like these. First, the constant of proportionality 1.25 means something important in the problem—it’s the price of 1 foot of rope. Because of that, the 1.25 could be easily used to compute the price of any length of rope. If no students bring it up, point out that the equation $y = 1.25x$ could be used to relate any length of rope, x, to its price, y.

3.3 Swimming, Manufacturing, and Painting

10 minutes

In this activity students use proportional relationships to solve problems. Although students might remember a few of the problems from previous lessons, students are asked to answer different questions. Students first need to recognize that the situation involves a proportional relationship and then use that knowledge to solve the problems. Notice that the problems decrease in the amount of scaffolding in order to build confidence for students as they work through them. Look out for students who have a systematic way of approaching these problems, and ask them to share their strategy during the discussion.

Addressing

- 7.RP.A.2

Instructional Routines

- MLR3: Clarify, Critique, Correct

Launch

Give students 3–5 minutes of quiet work time, followed by whole-class discussion.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organization and problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organization; Attention

Anticipated Misconceptions

Some students may struggle to continue working as the scaffolding is decreased. Consider using these questions to prompt students:

- What are the two associated quantities in this problem?
- How many quarts of blue paint are needed for 1 quart of white paint?

Student Task Statement

1. Tyler swims at a constant speed, 5 meters every 4 seconds. How long does it take him to swim 114 meters?

distance (meters)	time (seconds)
5	4
114	

2. A factory produces 3 bottles of sparkling water for every 8 bottles of plain water. How many bottles of sparkling water does the company produce when it produces 600 bottles of plain water?

number of bottles of sparkling water	number of bottles of plain water

3. A certain shade of light blue paint is made by mixing $1\frac{1}{2}$ quarts of blue paint with 5 quarts of white paint. How much white paint would you need to mix with 4 quarts of blue paint?

4. For each of the previous three situations, write an equation to represent the proportional relationship.

Student Response

1. 91.2 seconds or equivalent. Sample reasoning: Tyler swims 1 meters in 0.8 seconds because $4 \div 5 = 0.8$. It takes him 91.2 seconds to swim 114 meters, because $114 \cdot 0.8 = 91.2$

distance (meters)	time (seconds)
5	4
114	91.2

2. 225. Sample reasoning: The factory produces 0.375 of a bottle of sparkling water per bottle of plain water because $3 \div 8 = 0.375$. The factory produces 225 bottles of sparkling water when it produces 600 bottles of plain water, because $600 \cdot 0.375 = 225$.

number of bottles of sparkling water	number of bottles of plain water
3	8
225	600

3. $13\frac{1}{3}$ or equivalent. Sample reasoning: There are $3\frac{1}{3}$ quarts of white paint per quart of blue paint, because $5 \div 1\frac{1}{2} = 3\frac{1}{3}$. So you would need to mix $13\frac{1}{3}$ quarts of white paint with 4 quarts of blue paint, because $4 \cdot 3\frac{1}{3} = 4 \cdot \frac{10}{3} = 13\frac{1}{3}$.

blue paint (quarts)	white paint (quarts)
$1\frac{1}{2}$	5
4	$13\frac{1}{3}$

4. Answers vary. Sample response:
    - $t = \frac{4}{5}d$
    - $p = \frac{8}{3}s$
    - $w = \frac{10}{3}b$

Are You Ready for More?

Different nerve signals travel at different speeds.

- Pressure and touch signals travel about 250 feet per second.
- Dull pain signals travel about 2 feet per second.

1. How long does it take you to feel an ant crawling on your foot?
2. How much longer does it take to feel a dull ache in your foot?

Student Response

Answers vary. Sample response: The distance between your foot and your brain depends on how tall you are. If you are 5.5 feet tall, then:

1. It takes about $5.5 \div 250 = 0.022$ seconds for the signal to reach your brain.
2. It takes about $5.5 \div 2 = 2.75$ seconds for the pain signal to reach your brain.

Activity Synthesis

Students may have used the same method for each problem. For this reason, it might not be necessary to go over every problem. For the second question, you might select 2 different students who used the different methods to share their strategy. For the third question, ask students to share how they knew which quantities to put into the table. You might also ask:

- Does it matter which heading goes in which column?
- Do you get a different answer if you switch them? Why or why not?
- Would your strategy change if you switched them? Why or why not?

Ask students where they got stuck and what helped them to move through the hard parts. (Making a table; using the given information to figure out new information.)

Access for English Language Learners

Writing: MLR3 Clarify, Critique, Correct. Present an incorrect response to the question about mixing 4 quarts of blue paint. For example, "Since there are 0.3 quarts of blue paint to white paint, you need 1.2 quarts of white paint." Prompt students to identify the error (e.g., ask students, "Do you agree with the statement? Why or why not?"), and then write a correct version. This helps students evaluate, and improve on, the written mathematical arguments of others.
Design Principle(s): Maximize meta-awareness

3.4 Finishing the Race and More Orange Juice

Optional: 10 minutes
The purpose of this activity is to give students an opportunity to solve proportional relationships with fractions without the scaffolded support as given in the previous activity (MP1). If students get stuck, here are some questions to ask:

- How many miles do they run in one hour?
- How many cups of orange juice concentrate are needed for one cup of water?

Monitor for students who are using the constant of proportionality strategy they learned in an earlier activity. These students should be asked to share during the discussion.

Addressing

- 7.RP.A.1

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

- MLR1: Stronger and Clearer Each Time

Launch

Give students 3–5 minutes of quiet work time followed by whole-class discussion.

If time is limited, pick one of the problems to talk about during the discussion.

Access for Students with Disabilities

Representation: Internalize Comprehension. Represent the same information through different modalities by using a table of equivalent ratios. If students are unsure where to begin, suggest that they draw a table of equivalent ratios to help organize the information provided.
Supports accessibility for: Conceptual processing; Visual-spatial processing

Access for English Language Learners

Writing, Speaking: MLR1 Stronger and Clearer Each Time. To help students refine their explanations for the second question, listeners should press for details and clarity as appropriate based on what each speaker produces. Provide students with prompts for feedback that will help individuals strengthen their ideas and clarify their language (e.g., "How did you use the constant of proportionality?" "What steps did you take to find the amount of orange juice they each need?" "How do you know your answer is correct?" etc.). Students can borrow ideas and language from each partner to strengthen their final product.
Design Principle(s): Optimize output (for explanation)

Student Task Statement

1. Lin runs $2\frac{3}{4}$ miles in $\frac{2}{5}$ of an hour. Tyler runs $8\frac{2}{3}$ miles in $\frac{4}{3}$ of an hour. How long does it take each of them to run 10 miles at that rate?

2. Priya mixes $2\frac{1}{2}$ cups of water with $\frac{1}{3}$ cup of orange juice concentrate. Diego mixes $1\frac{2}{3}$ cups of water with $\frac{1}{4}$ cup orange juice concentrate. How much concentrate should each of them mix with 100 cups of water to make juice that tastes the same as their original recipe? Explain or show your reasoning.

Student Response

1. Lin takes $1\frac{5}{11}$ hours. Tyler takes $1\frac{7}{13}$ hours. Lin takes $\frac{8}{55}$ of an hour to run each mile because $\frac{2}{5} \div 2\frac{3}{4} = \frac{8}{55}$. Lin takes $1\frac{5}{11}$ of an hour to run 10 miles because $10 \cdot \frac{8}{55} = 1\frac{5}{11}$. Tyler takes $\frac{2}{13}$ of an hour to run each mile because $\frac{4}{3} \div 8\frac{2}{3} = \frac{2}{13}$. Tyler takes $1\frac{7}{13}$ of an hour to run 10 miles because $10 \cdot \frac{2}{13} = 1\frac{7}{13}$.

2. ◦ Priya should use $13\frac{1}{3}$ cups of concentrate. Sample explanation: Priya needs to make 40 batches of her recipe, because $2\frac{1}{2} \cdot 40 = 100$. This means she needs to use $40 \cdot \frac{1}{3}$, or $\frac{40}{3}$ cups of orange juice concentrate.

   ◦ Diego should use 15 cups of orange juice concentrate. Sample explanation: Diego uses $\frac{3}{20}$ cups of concentrate per cup of water, because $\frac{1}{4} \div \frac{5}{3} = \frac{3}{20}$. For 100 cups of water, he would need to use $100 \cdot \frac{3}{20}$, or 15 cups of orange juice concentrate.

Activity Synthesis

The purpose of this discussion is for students to share how they solved proportion problems involving fractions. Select previously identified students to share their method for finding a solution. Ask students to describe why they chose to calculate the constant of proportionality for this problem and how that helped them with finding the solution.

Lesson Synthesis

In this lesson, we worked efficiently with tables.

"How can we use a table that only has two rows to solve a problem about a proportional relationship?" (Calculate either a scale factor or the constant of proportionality, use this as a multiplier.)

3.5 Walnuts in Bulk

Cool Down: 5 minutes

Addressing

- 7.RP.A.1

Student Task Statement

It costs \$3.45 to buy $\frac{3}{4}$ lb of chopped walnuts. How much would it cost to purchase 7.5 lbs of walnuts? Explain or show your reasoning.

Student Response

\$34.50. Sample explanation: It costs 10 times as much to buy 7.5 lbs of walnuts as to buy $\frac{3}{4}$ lbs of walnuts since $\frac{3}{4} \cdot 10 = 7.5$. It costs \$ because $3.45 \cdot 10 = 34.50$.

Student Lesson Summary

If we identify two quantities in a problem and one is proportional to the other, then we can calculate the constant of proportionality and use it to answer other questions about the situation. For example, Andre runs at a constant speed, 5 meters every 2 seconds. How long does it take him to run 91 meters at this rate?

In this problem there are two quantities, time (in seconds) and distance (in meters). Since Andre is running at a constant speed, time is proportional to distance. We can make a table with distance and time as column headers and fill in the given information.

distance (meters)	time (seconds)
5	2
91	

To find the value in the right column, we multiply the value in the left column by $\frac{2}{5}$ because $\frac{2}{5} \cdot 5 = 2$. This means that it takes Andre $\frac{2}{5}$ seconds to run one meter.

At this rate, it would take Andre $\frac{2}{5} \cdot 91 = \frac{182}{5}$, or 36.4 seconds to walk 91 meters. In general, if t is the time it takes to walk d meters at that pace, then $t = \frac{2}{5}d$.

Lesson 3 Practice Problems

Problem 1

Statement

It takes an ant farm 3 days to consume $\frac{1}{2}$ of an apple. At that rate, in how many days will the ant farm consume 3 apples?

Solution

18 days

Problem 2

Statement

To make a shade of paint called jasper green, mix 4 quarts of green paint with $\frac{2}{3}$ cups of black paint. How much green paint should be mixed with 4 cups of black paint to make jasper green?

Solution

24 quarts

Problem 3

Statement

An airplane is flying from New York City to Los Angeles. The distance it travels in miles, d, is related to the time in seconds, t, by the equation $d = 0.15t$.

a. How fast is it flying? Be sure to include the units.

b. How far will it travel in 30 seconds?

c. How long will it take to go 12.75 miles?

Solution

a. It is traveling at 0.15 miles per second.

b. It will travel 4.5 miles in 30 seconds.

c. It will take 85 seconds to travel 12.75 miles.

Problem 4

Statement

A grocer can buy strawberries for $1.38 per pound.

a. Write an equation relating c, the cost, and p, the pounds of strawberries.

b. A strawberry order cost $241.50. How many pounds did the grocer order?

Solution

a. $c = 1.38p$

b. 175 pounds

Problem 5

Statement

Crater Lake in Oregon is shaped like a circle with a diameter of about 5.5 miles.

a. How far is it around the perimeter of Crater Lake?

b. What is the area of the surface of Crater Lake?

Solution

a. About 17 miles (5.5π)

b. About 24 square miles ($\pi \cdot 2.75^2$)

(From Unit 3, Lesson 10.)

Problem 6

Statement

A 50-centimeter piece of wire is bent into a circle. What is the area of this circle?

Solution

$\frac{625}{\pi}$ or about 199 cm^2

(From Unit 3, Lesson 8.)

Problem 7

Statement

Suppose Quadrilaterals A and B are both squares. Are A and B necessarily scaled copies of one another? Explain.

Solution

Yes. Since all four side lengths of a square are the same, whatever scale factor works to scale one edge of A to an edge of B takes all edges of A to all edges of B. Since scaling a square gives another square, B is a scaled copy of A.

(From Unit 1, Lesson 2.)