Lesson 4: Half as Much Again

Goals

- Apply the distributive property to generate algebraic expressions that represent a situation involving adding or subtracting a fraction of the initial value, and explain (orally) the reasoning.
- Coordinate tables, equations, tape diagrams, and verbal descriptions that represent a relationship involving adding or subtracting a fraction of the initial value.
- Generalize a process for finding the value that is “half as much again,” and justify (orally and in writing) why this can be abstracted as $\frac{3}{2}x$ or equivalent.

Learning Targets

- I can use the distributive property to rewrite an expression like $x + \frac{1}{2}x$ as $(1 + \frac{1}{2})x$.
- I understand that “half as much again” and “multiply by $\frac{3}{2}$” mean the same thing.

Lesson Narrative

In this lesson students see how to use the distributive property to write a compact expression for situations where one quantity is described in relation to another quantity in language such as "half as much again" and "one third more than." If y is half as much again as x, then $y = x + \frac{1}{2}x$. Using the distributive property, this can be written as $y = (1\frac{1}{2})x$. Students apply this sort of reasoning to various situations. A warm-up activity activates their prior knowledge of using the distributive property to write equivalent expressions. When students look for opportunities to use the distributive property to write equations in a simpler way, they are engaging in MP7.

In the next lesson they will consider similar situations involving fractions expressed as decimals. These two lessons prepare them for later study of situations involving percent increase and percent decrease.

Alignments

Building On

- 6.EE.A.3: Apply the properties of operations to generate equivalent expressions. For example, apply the distributive property to the expression $3(2 + x)$ to produce the equivalent expression $6 + 3x$; apply the distributive property to the expression $24x + 18y$ to produce the equivalent expression $6(4x + 3y)$; apply properties of operations to $y + y + y$ to produce the equivalent expression $3y$.
- 6.RP.A.3: Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

- 7.RP.A.1: Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks $1/2$ mile in each $1/4$ hour, compute the unit rate as the complex fraction $\frac{1/2}{1/4}$ miles per hour, equivalently 2 miles per hour.

Addressing

- 7.RP.A.2: Recognize and represent proportional relationships between quantities.

Building Towards

- 7.EE.A.1: Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.
- 7.RP.A.3: Use proportional relationships to solve multistep ratio and percent problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR7: Compare and Connect
- MLR8: Discussion Supports
- Notice and Wonder

Required Preparation

Print and cut up slips from the Representations of Proportional Relationships Card Sort blackline master. Prepare 1 copy for every 2 students. These can be re-used if you have more than one class. Consider making a few extra copies that are not cut up to serve as an answer key.

> ### Student Learning Goals
>
> Let's use fractions to describe increases and decreases.

4.1 Notice and Wonder: Tape Diagrams

Warm Up: 5 minutes

The purpose of this warm-up is to elicit the idea that there are different ways to represent a value, which will be useful when students apply the distributive property to write expressions in a later activity. While students may notice and wonder many things about these images, the similarities and differences between the two tape diagrams are the important discussion points.

Building On

- 6.EE.A.3

Building Towards

- 7.RP.A.3

Instructional Routines

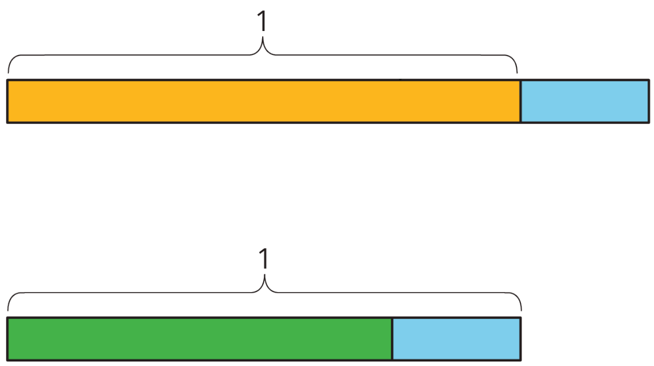
- Notice and Wonder

Launch

Arrange students in groups of 2. Tell students that they will look at an image, and their job is to think of at least one thing they notice and at least one thing they wonder. Display the image for all to see. Ask students to give a signal when they have noticed or wondered about something. Give students 1 minute of quiet think time, and then 1 minute to discuss the things they notice with their partner, followed by a whole-class discussion.

Student Task Statement

What do you notice? What do you wonder?



Student Response

Answers vary. Samples responses:

I notice:

- In each diagram I am putting together two rectangles of different lengths.
- Each diagram has a blue rectangle and one other rectangle.
- The blue rectangles in each diagram look to be the same size.
- The yellow rectangle is the same length as the green and blue rectangles put together.
- The green rectangle looks to be three times as long as the blue rectangle.
- The yellow rectangle looks to be four times as long as the blue rectangle.
- The green rectangle looks to be three-fourths as long as the yellow rectangle.

I wonder:

- How many blue rectangles would it take to cover the green or yellow rectangle?
- What fraction does the blue rectangle represent?
- How long is the total length of the the first diagram?
- What is the missing length in the second diagram?
- What situations do these diagrams represent?

Activity Synthesis

Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If possible, record the relevant reasoning on or near the image. After each response, ask the class if they agree or disagree and to explain alternative ways of thinking, referring back to the images each time. If the total length of the first diagram does not come up during the conversation, ask students to discuss this idea.

4.2 Walking Half as Much Again

10 minutes

In this activity, students find patterns in situations to connect to the distributive property (MP7). These patterns build understanding of the equations $x + 0.5x = (1 + 0.5)x = 1.5x$ or $x + \frac{1}{2}x = (1 + \frac{1}{2})x = 1\frac{1}{2}x$. Students should see that the expressions are all representations of the same thing. Students learn that multiplying a number by $\frac{1}{2}$ and adding that product to the original numbers is the same as multiplying by 1.5. This idea is extended to percents in the following activity.

As students work on the task, monitor the reasoning they come up with for the question comparing Mai and Kiran's representations (MP3). Identify students who agree with both Mai and Kiran; these students should be chosen to share during the discussion.

Building On

- 6.EE.A.3

Building Towards

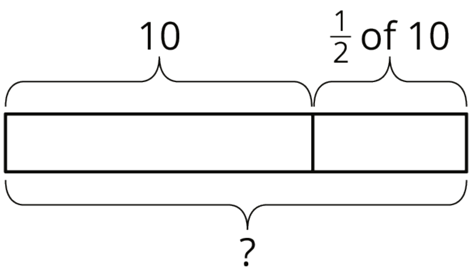
- 7.EE.A.1
- 7.RP.A.3

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR8: Discussion Supports

Launch

Demonstrate, or ask students to demonstrate, walking a certain distance and then walking “half as much again”. Ask students how they could draw a tape diagram to represent the first situation. For example, they could draw something like this.



Give students 5 minutes of quiet work time followed by partner and then whole-class discussion.

Anticipated Misconceptions

Students might struggle coming up with an expression in terms of x. Ask students to describe how you would calculate it in words, then see if they can use that to write the expression.

When comparing Kiran and Mai's equations, even though there is no coefficient written in front of the x in Mai's equation, x is equivalent to $1x$.

Student Task Statement

1. Complete the table to show the total distance walked in each case.

   a. Jada’s pet turtle walked 10 feet, and then half that length again.

   b. Jada’s baby brother walked 3 feet, and then half that length again.

   c. Jada’s hamster walked 4.5 feet, and then half that length again.

   d. Jada’s robot walked 1 foot, and then half that length again.

   e. A person walked x feet and then half that length again.

initial distance	total distance
10	
3	
4.5	
1	
x	

2. Explain how you computed the total distance in each case.

3. Two students each wrote an equation to represent the relationship between the initial distance walked (x) and the total distance walked (y).

   - Mai wrote $y = x + \frac{1}{2}x$.
   - Kiran wrote $y = \frac{3}{2}x$.

   Do you agree with either of them? Explain your reasoning.

Student Response

1.

initial distance	total distance
10	15
3	4.5
4.5	6.75
1	1.5
x	$x + \frac{1}{2}x$

2. Answers vary. Sample response: I took the half of the initial distance walked and added it to the initial distance walked to get the total distance walked.
3. They are both correct. Explanations vary. Sample explanations:
   - The quotient of total distance walked and the corresponding initial distance walked is constant (it's 1.5).
   - The total distance walked is always 1.5 times the initial distance walked.
   - I agree with Kiran because $10 \cdot 1.5 = 15$ and that works for every other entry in the table.

Are You Ready for More?

Zeno jumped 8 meters. Then he jumped half as far again (4 meters). Then he jumped half as far again (2 meters). So after 3 jumps, he was $8 + 4 + 2 = 14$ meters from his starting place.

1. Zeno kept jumping half as far again. How far would he be after 4 jumps? 5 jumps? 6 jumps?
2. Before he started jumping, Zeno put a mark on the floor that was exactly 16 meters from his starting place. How close can Zeno get to the mark if he keeps jumping half as far again?
3. If you enjoyed thinking about this problem, consider researching Zeno's Paradox.

Student Response

1. 15, $15\frac{1}{2}$, $15\frac{3}{4}$
2. He can get as close as he wants, because he can always cut the distance between his current position and the mark in half.
3. No response expected.

Activity Synthesis

Ask selected students to share who they agree with (Mai or Kiran) and have them explain how they know. Highlight instances when students are reasoning with the distributive property, and encourage them to identify it by name. Suggested questions:

- "Are Mai and Kiran's equations equivalent?"
- "What is the same and what is different about the two equations?"
- "How does each equation represent the "one and a half" relationship between the quantities?"
- "How else could you write an equation to represent the relationship?" ($y = (1 + 0.5)x$, $y = \frac{3}{2}x$, $y = 1x + 0.5x$) Students should see that the equations express the same relationship and understand why.

Access for Students with Disabilities

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation such as "First, I _____ because...", "I noticed _____ so I...", "Why did you...?", "I agree/disagree because...."
Supports accessibility for: Language; Social-emotional skills

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Access for English Language Learners

Speaking, Listening: MLR8 Discussion Supports. Use this routine to support whole-class discussion. After selected students share their explanations for who they agree with (Mai or Kiran). Call on students to restate and/or revoice the explanations presented using mathematical language (e.g., distributive property, equivalent equation, etc.). Consider giving students time to practice restating what they heard to a partner, before selecting one or two students to share with the class. This will provide more students with an opportunity to produce language to explain whether two equations are equivalent.
Design Principle(s): Support sense-making; Maximize meta-awareness

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4.3 More and Less

10 minutes

The purpose of this activity is to connect various representations of proportional relationships including images, equations, and descriptions. Students first match descriptions of a proportional relationship, involving a variable x, to tape diagrams that represent the situations. Then they create equations that describe the proportion using only variables. Finally, they apply their understanding to create their own scenarios and describe a proportional relationship, which is already represented in the form of a tape diagram.

The focus of the discussion following the activity is on the numbers used to go from between significant quantities in a proportional relationship. For example, what numbers are relevant as we go from the quantity of blueberries eaten by Han to the quantity eaten by Mai?

Building On

- 6.RP.A.3

Building Towards

- 7.RP.A.3

Instructional Routines

- MLR7: Compare and Connect

Launch

Arrange students in groups of 2. Give students 1--2 minutes of quiet think time followed by partner discussion on the first problem. Then give students 3--5 minutes to complete the problems. Follow with whole-class discussion.

Access for Students with Disabilities

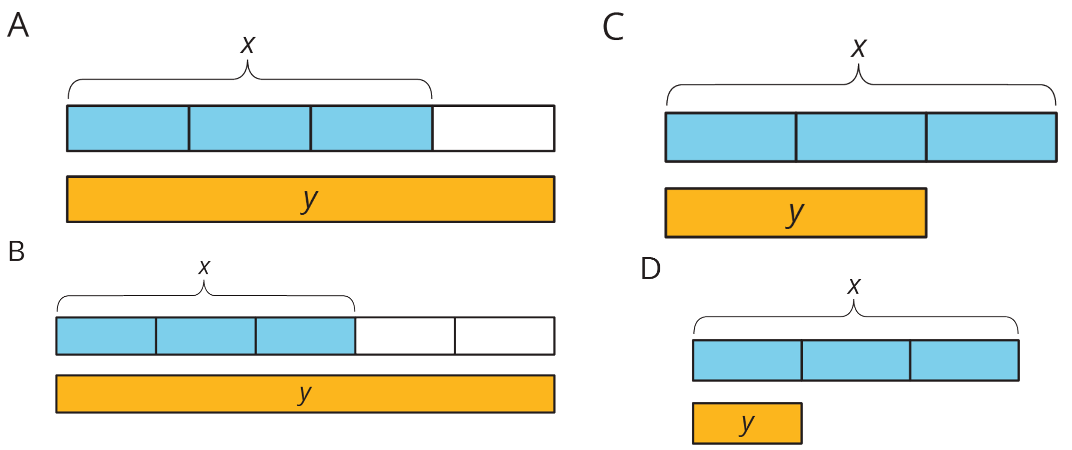
Representation: Internalize Comprehension. Demonstrate and encourage students to use color coding and annotations to highlight connections between representations in a problem. For example, use the same color to show how $\frac{1}{3}$ is represented in the situation, the tape diagram, and in the corresponding equation.
Supports accessibility for: Visual-spatial processing

Anticipated Misconceptions

Students may match the equation $y = \frac{2}{3}x$ with the situation “Mai biked x miles, and Han biked $\frac{2}{3}$ more than that.” Ask them who biked farther, Han or Mai? The equation they choose must result in Han biking a greater distance than Mai.

Student Task Statement

1. Match each situation with a diagram. A diagram may not have a match.



    - Han ate x ounces of blueberries. Mai ate $\frac{1}{3}$ less than that.
    - Mai biked x miles. Han biked $\frac{2}{3}$ more than that.
    - Han bought x pounds of apples. Mai bought $\frac{2}{3}$ of that.

2. For each diagram, write an equation that represents the relationship between x and y.
    a. Diagram A:
    b. Diagram B:
    c. Diagram C:
    d. Diagram D:

3. Write a story for one of the diagrams that doesn't have a match.

Student Response

1.
    - C. Han ate x ounces of blueberries. Mai ate $\frac{1}{3}$ less than that. Diagram C shows that y is less than x by one third.
    - B. Mai biked x miles. Han biked $\frac{2}{3}$ more than that. Diagram B shows that y is more than x by two-thirds.
    - C. Han bought x pounds of apples. Mai bought $\frac{2}{3}$ of that. Diagram C shows that y consists of two-thirds of x.

2. A: $y = \frac{4}{3}x$, B: $y = \frac{5}{3}x$, C: $y = \frac{2}{3}x$, D: $y = \frac{1}{3}x$ (or equivalent)

3. Answers vary. Sample responses: Mai slept x hours. Han slept $\frac{1}{3}$ more than that (for diagram A). Han has x quarters. Mai has $\frac{1}{3}$ of that (for diagram D).

Activity Synthesis

After students complete the task, ask students to share a few of their arguments for the matches they came up with (not all need to be shared). Questions you could ask them as they share:

- What is different about the numbers in the description and the numbers in the equation?
- How is the number in the equation related to the number in the description?
- What is different between a fractional amount more than the original versus less than the original?

Access for English Language Learners

Representing, Listening: MLR7 Compare and Connect. As students share their explanations for the matches they came up with with the class, call students' attention to the different ways quantities are represented in the descriptions, tape diagrams and equations. Wherever possible, amplify student words and actions that describe the correspondence between specific features of one mathematical representation with a specific feature of another representation. This will help students interpret what is communicated by each type of representation.
Design Principle(s): Maximize meta-awareness; Support sense-making

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4.4 Card Sort: Representations of Proportional Relationships

Optional: 10 minutes
In this activity, students receive a set of cards that have 3 different representations of proportional relationships (table, equation, description). The purpose of this activity is for students to interpret wording like "y is $\frac{1}{3}$ more than x" and figure out how that relationship can be expressed using different mathematical representations. Students may initially think that the wording means $y = \frac{1}{3}x$, however the numbers in the table representation will help them realize that this is not true. They are engaging in this reasoning here in preparation for upcoming work on percent increase and decrease.

They will work with their partner to match the different representations with each other (each set should contain a table, equation and description) and explain their reasoning (MP3).

There are 8 groups of 3 to match up. Here is an example of one of the groups of 3:

Noah ate x ounces of blueberries, and Elena ate $\frac{1}{3}$ less than that.	$y = \frac{2}{3}x$	<table><tr><td>x</td><td>y</td></tr><tr><td>3</td><td>2</td></tr><tr><td>18</td><td>12</td></tr></table>

Building On

- 7.RP.A.1

Addressing

- 7.RP.A.2

Building Towards

- 7.RP.A.3

Launch

Demonstrate how to set up and do the matching activity. Choose a student to be your partner. Mix up the rest of the cards and place them face-up. Point out that each card contains a description, table, or equation. Select one of each style of card and then explain to your partner why you think the cards do or do not match. Demonstrate productive ways to agree or disagree (e.g., by explaining your mathematical thinking, asking clarifying questions, etc.).

Arrange students in groups of 2. Give each group pre-printed cut-up slips for matching. Place two copies of uncut blackline masters in envelopes to serve as answer keys.

Access for Students with Disabilities

Representation: Internalize Comprehension. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity by beginning with fewer cards. For example, give students a subset of the cards to start with and introduce the remaining cards once students have completed their initial set of matches.
Supports accessibility for: Conceptual processing; Organization

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Anticipated Misconceptions

Students may match the equation $y = \frac{2}{3}x$ with the situation “Elena biked x miles, and Noah biked $\frac{2}{3}$ more than that.” Ask them who biked farther, Noah or Elena? The equation they choose must result in Noah biking a greater distance than Elena.

Student Task Statement

Your teacher will give you a set of cards that have proportional relationships represented three different ways: as descriptions, equations, and tables. Mix up the cards and place them all face-up.

1. Take turns with a partner to match a description with an equation and a table.
    a. For each match you find, explain to your partner how you know it's a match.
    b. For each match your partner finds, listen carefully to their explanation, and if you disagree, explain your thinking.
2. When you agree on all of the matches, check your answers with the answer key. If there are any errors, discuss why and revise your matches.

Student Response

The blackline master shows the correct matches.

Activity Synthesis

After students complete the task, ask students to share a few of their arguments for the matches they came up with (not all need to be shared). Questions you could ask them as they share:

- What is different about the numbers in the description and the numbers in the equation?
- How is the number in the equation related to the number in the description?
- What is different between a fractional amount more than the original versus less than the original?

Lesson Synthesis

Students should understand the role the distributive property plays in making calculations more efficient. Ask students:

- "Give examples of how we can use the distributive property to create equivalent expressions that make it easier for us to calculate an amount plus (or minus) a fraction of that amount." (e.g. $x + \frac{1}{2}x = 1\frac{1}{2}x$)
- "What does this look like in different representations?" (refer to the card sort examples)

4.5 Fruit Snacks and Skating

Cool Down: 5 minutes

If students only write expressions, like $\frac{1}{4}x$ and $\frac{8}{5}x$, instead of equations, then they have shown they achieved the goal of the lesson. It's not necessary or desirable to hold students accountable for correctly interpreting *expressions* versus *equations* at this time.

Building Towards

- 7.RP.A.3

Student Task Statement

1. Tyler ate x fruit snacks, and Han ate $\frac{3}{4}$ less than that. Write an equation to represent the relationship between the number Tyler ate (x) and the number Han ate (y).

2. Mai skated x miles, and Clare skated $\frac{3}{5}$ farther than that. Write an equation to represent the relationship between the distance Mai skated (x) and the distance Clare skated (y).

Student Response

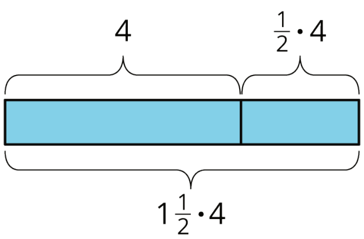
1. $y = \frac{1}{4}x$ (or equivalent). Han ate $\frac{3}{4}x$ less than the number of fruit snacks Tyler ate. Han ate $\frac{1}{4}x$ fruit snacks because $x - \frac{3}{4}x = \frac{1}{4}x$.

2. $y = \frac{8}{5}x$ (or equivalent). Clare skated $\frac{3}{5}x$ farther than the number of miles Mai skated. Clare skated $\frac{8}{5}x$ miles because $x + \frac{3}{5}x = \frac{8}{5}x$.

Student Lesson Summary

Using the distributive property provides a shortcut for calculating the final amount in situations that involve adding or subtracting a fraction of the original amount.

For example, one day Clare runs 4 miles. The next day, she plans to run that same distance plus half as much again. How far does she plan to run the next day?



Tomorrow she will run 4 miles plus $\frac{1}{2}$ of 4 miles. We can use the distributive property to find this in one step: $1 \cdot 4 + \frac{1}{2} \cdot 4 = \left(1 + \frac{1}{2}\right) \cdot 4$

Clare plans to run $1\frac{1}{2} \cdot 4$, or 6 miles.

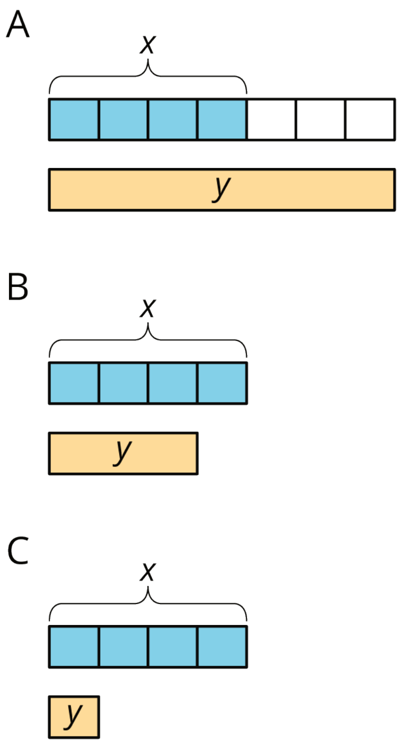
This works when we decrease by a fraction, too. If Tyler spent x dollars on a new shirt, and Noah spent $\frac{1}{3}$ less than Tyler, then Noah spent $\frac{2}{3}x$ dollars since $x - \frac{1}{3}x = \frac{2}{3}x$.

Glossary

- tape diagram

Lesson 4 Practice Problems
Problem 1
Statement

Match each situation with a diagram.



A. Diagram A

B. Diagram B

C. Diagram C

1. Diego drank x ounces of juice. Lin drank $\frac{1}{4}$ less than that.
2. Lin ran x miles. Diego ran $\frac{3}{4}$ more than that.
3. Diego bought x pounds of almonds. Lin bought $\frac{1}{4}$ of that.

Solution

- A: 2
- B: 1
- C: 3

Problem 2

Statement

Elena walked 12 miles. Then she walked $\frac{1}{4}$ that distance. How far did she walk all together? Select **all** that apply.

A. $12 + \frac{1}{4}$

B. $12 \cdot \frac{1}{4}$

C. $12 + \frac{1}{4} \cdot 12$

D. $12\left(1 + \frac{1}{4}\right)$

E. $12 \cdot \frac{3}{4}$

F. $12 \cdot \frac{5}{4}$

Solution

["C", "D", "F"]

Problem 3

Statement

Write a story that can be represented by the equation $y = x + \frac{1}{4}x$.

Solution

Answers vary. Sample response: Andre slept x hours. Diego slept $\frac{1}{4}$ more than that.

Problem 4

Statement

Select **all** ratios that are equivalent to $4 : 5$.

A. $2 : 2.5$

B. $2 : 3$

C. $3 : 3.75$

D. $7 : 8$

E. $8 : 10$

F. $14 : 27.5$

Solution

["A", "C", "E"]

(From Unit 4, Lesson 1.)

Problem 5

Statement

Jada is making circular birthday invitations for her friends. The diameter of the circle is 12 cm. She bought 180 cm of ribbon to glue around the edge of each invitation. How many invitations can she make?

Solution

Each card needs 12π or about 37.7 cm of ribbon. She has enough ribbon for 4 cards since $180 \div 37.7 \approx 4.77$.

(From Unit 3, Lesson 10.)