Lesson 3: Rectangle Madness

Goals

- Coordinate diagrams and expressions involving equivalent fractions.
- Interpret and create diagrams involving a rectangle decomposed into squares.
- Recognize that decomposing rectangles into squares is a geometric way to determine the greatest common factor of two numbers.

Lesson Narrative

This lesson is optional. In this exploration in pure mathematics, students tackle a series of activities that explore the relationship between the greatest common factor of two numbers and related fractions using a geometric representation. The activities in this lesson build on each other, providing students an opportunity to express the relationship between the greatest common factor of two numbers and related fractions through repeated reasoning (MP8). Thus, the activities should be done in order. Doing all of the activities would take more than a single class period—possibly as many as four. It is up to the teacher how much time to spend on this topic. It is not necessary to do the entire set of problems to get some benefit from the activities in this lesson, although more connections are made the farther one gets. As with all lessons in this unit, all related standards have been addressed in prior units; this lesson provides an *optional* opportunity to go more deeply and make connections between domains.

Alignments

Building On

- 4.MD.A.3: Apply the area and perimeter formulas for rectangles in real world and mathematical problems. For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.
- 5.NF.B.3: Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. For example, interpret $3/4$ as the result of dividing 3 by 4, noting that $3/4$ multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size $3/4$. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?
- 6.NS.B.4: Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express $36 + 8$ as $4(9 + 2)$.

Addressing

- 6.NS.A: Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

Instructional Routines

- MLR7: Compare and Connect
- MLR8: Discussion Supports

Required Materials

Graph paper

> ### Student Learning Goals
>
> Let’s cut up rectangles.

3.1 Squares in Rectangles

Optional: 15 minutes
This first activity helps students understand the geometric process that they use in later activities to connect the greatest common factor with related fractions. The first question helps students focus on the impact on side lengths of decomposing a rectangle into smaller rectangles. The second question has students analyze a rectangle that has been decomposed into squares (MP7). The third question has students decompose a rectangle into squares themselves. In the next activity, students relate this process to greatest common factors and fractions.

Building On

- 4.MD.A.3

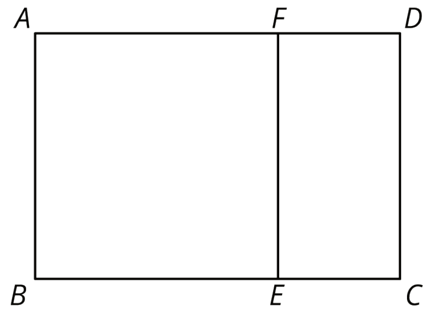
Instructional Routines

- MLR7: Compare and Connect

Launch

Give 5 minutes of quiet think time followed by whole-class discussion. Consider doing a notice and wonder with the first diagram to help students make sense of the way the vertices are named.

> ### Student Task Statement
>
> 1. Rectangle $ABCD$ is not a square. Rectangle $ABEF$ is a square.



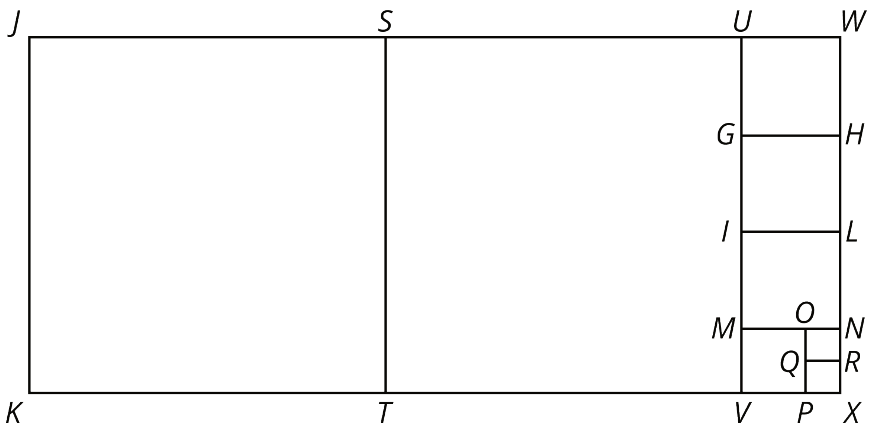
a. Suppose segment AF were 5 units long and segment FD were 2 units long. How long would segment AD be?

b. Suppose segment BC were 10 units long and segment BE were 6 units long. How long would segment EC be?

c. Suppose segment AF were 12 units long and segment FD were 5 units long. How long would segment FE be?

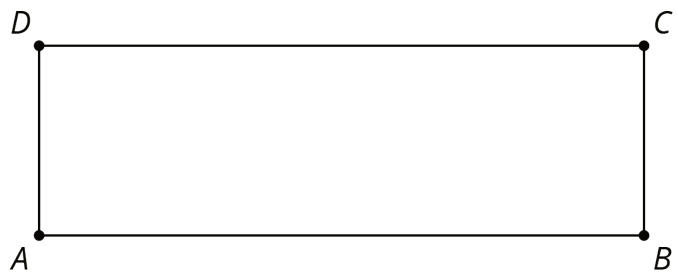
d. Suppose segment AD were 9 units long and segment AB were 5 units long. How long would segment FD be?

2. Rectangle $JKXW$ has been decomposed into squares.



Segment JK is 33 units long and segment JW is 75 units long. Find the areas of all of the squares in the diagram.

3. Rectangle $ABCD$ is 16 units by 5 units.



a. In the diagram, draw a line segment that decomposes $ABCD$ into two regions: a square that is the largest possible and a new rectangle.

b. Draw another line segment that decomposes the *new* rectangle into two regions: a square that is the largest possible and another new rectangle.

c. Keep going until rectangle $ABCD$ is entirely decomposed into squares.

d. List the side lengths of all the squares in your diagram.

Student Response

1. In rectangle $ABCD$:
    a. 7

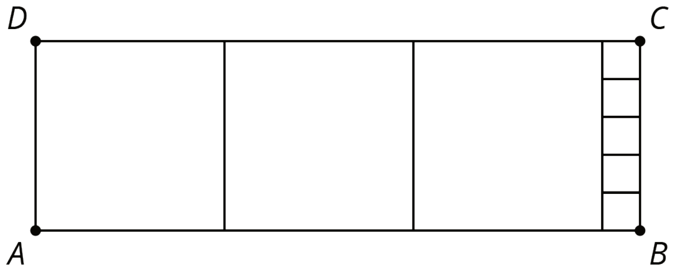
    b. 4

    c. 12

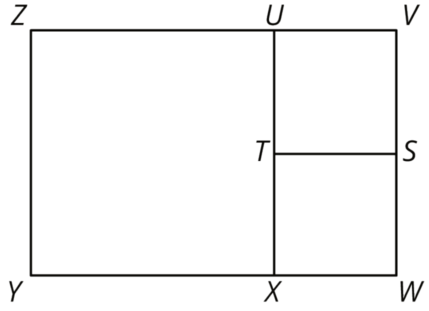
    d. 4

2. 1,089, 81, 36, and 9. The area of both $JSTK$ and $SUVT$ is 33^2, or 1,089 square units. The sides of $UWHG$, $GHLI$, and $ILNM$ are 9 units, so each of their areas is 81 square units. The side of $MOPV$ is 6 units, so its area is 36 square units. The sides of $ONRQ$ and $QRXP$ are 3 units, so their areas are 9 square units.

3. Here is the rectangle decomposed into squares. (There is more than one way to do it, but any approach will result in the same number of squares of the same size.) There are three 5-by-5 squares and five 1-by-1 squares.



Are You Ready for More?



1. The diagram shows that rectangle $VWYZ$ has been decomposed into three squares. What could the side lengths of this rectangle be?
2. How many different side lengths can you find for rectangle $VWYZ$?
3. What are some rules for possible side lengths of rectangle $VWYZ$?

Student Response

Answers vary. Sample responses:

1. Height is 2 units, width is 3 units.
2. Any number of distinct Rectangles $VWYZ$ could be created.
3. The length of line segment XW can be any number. Then the width of $VWYZ$ is 3 times the length of line segment XW, and the height of $VWYZ$ is 2 times the length of line segment XW.

Activity Synthesis

Ask students what difficulties they had and how they resolved them. Two insights are helpful when working with rectangles that have been partitioned into squares:

- Squares have four sides of the same length. If you know the length of one side of a square, then you know the lengths of the other three sides of the square.
- If a line segment of length x units is decomposed into two segments of length y units and z units, then $x = y + z$. This relationship can be expressed in other ways, like $x - z = y$.

Access for English Language Learners

Speaking: MLR7 Compare and Connect. Use this routine when students share their process for decomposing the rectangle for the final question. Ask students, to compare the different ways the rectangle was decomposed. For example, ask: "How are the decompositions of the rectangle the same? How are they different?" In this discussion, emphasize that the number of squares created by the different decompositions is the same, regardless of the process. These exchanges can strengthen students' mathematical language use and their ability to recognize patterns across geometric processes.
Design Principle(s): Maximize meta-awareness

3.2 More Rectangles, More Squares

Optional: 30 minutes
In this activity, students apply the geometric process that they saw in the last activity and are asked to make connections between the result and the greatest common factor of the side lengths of the original rectangle. They work on a sequence of similar problems, allowing them to see and begin to

articulate a pattern (MP8). Then they make connections between this pattern and fractions that are equivalent to the fraction made up of the side lengths of the original rectangle.

Building On

- 5.NF.B.3
- 6.NS.B.4

Instructional Routines

- MLR8: Discussion Supports

Launch

Arrange students in groups of 2. Students work on problems alone and check work with a partner.

Review fractions with a mini warm-up. Sample reasoning is shown. Drawing a diagram might help.

1. Write a fraction that is equal to the mixed number $3\frac{4}{5}$. (We can think of $3\frac{4}{5}$ as $\frac{15}{5} + \frac{4}{5}$, so it is equal to $\frac{19}{5}$.)
2. Write a mixed number that is equal to this fraction: $\frac{11}{4}$. (This is equal to $\frac{8}{4} + \frac{3}{4}$, so it is equal to $2\frac{3}{4}$.)

It is quicker to sketch the rectangles on blank paper, but some students may benefit from using graph paper to support entry into these problems.

If students did not do the previous activity the same day as this activity, remind them of the earlier work:

- Draw a rectangle that is 16 units by 5 units.
    a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until the diagram shows that your original rectangle has been decomposed into squares.
    b. How many squares of each size are there? (Three 5-by-5 squares and five 1-by-1 squares.)
    c. What is the side length of the smallest square? (1)

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge about the connection between fractions and decomposition of rectangles. Some students may benefit from a physical demonstration of how to draw line segments to decompose rectangles into squares. Invite students to engage in the process by offering suggested directions as you demonstrate.
Supports accessibility for: Visual-spatial processing; Organization

Anticipated Misconceptions

Students may not see any connections between the decomposition of the rectangles and the fraction problems. Encourage them to look at the number of squares in the decomposition.

Student Task Statement

1. Draw a rectangle that is 21 units by 6 units.
    a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until the diagram shows that your original rectangle has been entirely decomposed into squares.
    b. How many squares of each size are in your diagram?
    c. What is the side length of the smallest square?
2. Draw a rectangle that is 28 units by 12 units.
    a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until the diagram shows that your original rectangle has been decomposed into squares.
    b. How many squares of each size are in your diagram?
    c. What is the side length of the smallest square?
3. Write each of these fractions as a mixed number with the smallest possible numerator and denominator:
    a. $\frac{16}{5}$
    b. $\frac{21}{6}$
    c. $\frac{28}{12}$

4. What do the fraction problems have to do with the previous rectangle decomposition problems?

Student Response

1. 21-by-6 rectangle
    a. Diagrams vary.

    b. Three 6-by-6 squares and two 3-by-3 squares

    c. 3

2. 28-by-12 rectangle
    a. Diagrams vary.

    b. Two 12-by-12 squares and three 4-by-4 squares

    c. 4

3. Fractions as mixed numbers:
    a. $\frac{16}{5} = 3\frac{1}{5}$

    b. $\frac{21}{6} = 3\frac{3}{6} = 3\frac{1}{2}$

    c. $\frac{28}{12} = 2\frac{4}{12} = 2\frac{1}{3}$

4. Answers vary. Sample responses: The given fractions had the same numbers as the side lengths of the given rectangles. The mixed numbers included the numbers of squares. For example, the 28-by-12 rectangle was partitioned into 2 large squares and 3 small squares, and the associated mixed number was $2\frac{1}{3}$. The size of the smallest square is related to the fraction used to make the mixed number. For example, in the 28-by-12 rectangle, the smallest square had sides of length 4. To make the mixed number, $\frac{4}{12}$ is rewritten as $\frac{1}{3}$.

Activity Synthesis

Invite students to share some of their observations. Ask, "What connections do you see between the rectangle drawings and the fractions?"

At this point, it is sufficient for students to notice *some* connections between a partitioned rectangle and its associated fraction. They will have more opportunities to explore, so there's no need to make sure they notice *all* of the connections right now. Here are examples of things that it is possible to notice using the rectangles in this activity but might not be noticed until students see more examples:

- The given fractions had the same numbers as the side lengths of the given rectangles.
- The mixed numbers included the numbers of squares. For example, the 28-by-12 rectangle was partitioned into 2 large squares and 3 small squares, and the associated mixed number was $2\frac{1}{3}$.

- The size of the smallest square is related to the fraction used to make the mixed number. For example, in the 28-by-12 rectangle, the smallest square had sides of length 4. To make the mixed number, $\frac{4}{12}$ is rewritten as $\frac{1}{3}$.

Access for English Language Learners

Speaking: MLR8 Discussion Supports. Press for details in students' explanations by requesting that students challenge an idea, elaborate on an idea, or make explicit connections between the two representations (numerical and geometric). Encourage students to use the terms partition and decomposition in their explanations.
Design Principle(s): Support sense-making; Optimize output (for explanation)

3.3 Finding Equivalent Fractions

Optional: 30 minutes
In this activity, students make the connection between the fraction determined by the original rectangle and the resulting fraction more precise (MP6). The two rectangles taken together are designed to help students notice that decomposing rectangles is a geometric way to determine the greatest common factor of two numbers. (This is a geometric version of Euclid's algorithm for finding the greatest common factor.)

Building On

- 5.NF.B.3
- 6.NS.B.4

Launch

Arrange students in groups of 2. Provide access to graph paper. Students work on problems alone and compare work with a partner.

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge. Provide students with access to the two large rectangles prepared on graph paper. Encourage students to annotate diagrams with details to show how each value is represented. For example, number of squares in total, side length of the smallest square.
Supports accessibility for: Visual-spatial processing; Organization

Student Task Statement

1. Accurately draw a rectangle that is 9 units by 4 units.

a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. How many squares of each size are there?

c. What are the side lengths of the last square you drew?

d. Write $\frac{9}{4}$ as a mixed number.

2. Accurately draw a rectangle that is 27 units by 12 units.

    a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

    b. How many squares of each size are there?

    c. What are the side lengths of the last square you drew?

    d. Write $\frac{27}{12}$ as a mixed number.

    e. Compare the diagram you drew for this problem and the one for the previous problem. How are they the same? How are they different?

3. What is the greatest common factor of 9 and 4? What is the greatest common factor of 27 and 12? What does this have to do with your diagrams of decomposed rectangles?

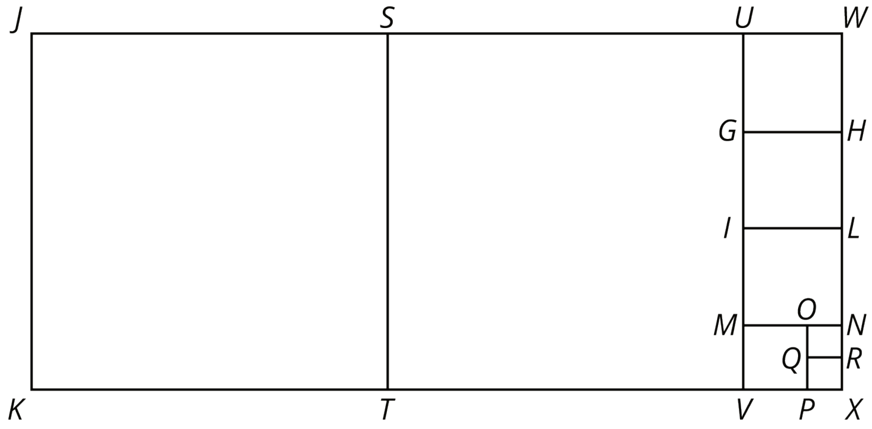
Student Response

1. 9-by-4 rectangle.
    a. Diagrams vary.
    b. Two 4-by-4 squares and four 1-by-1 squares
    c. 1 by 1
    d. $\frac{9}{4} = 2\frac{1}{4}$
2. 27-by-12 rectangle:
    a. Diagrams vary.
    b. Two 12-by-12 squares and four 3-by-3 squares
    c. 3 by 3
    d. $\frac{27}{12} = 2\frac{3}{12} = 2\frac{1}{4}$
    e. It is similar because there are 2 large squares and 4 small squares. It is different because the squares are not the same size. The larger rectangle partitioned into larger squares is a scaled up version of the smaller rectangle partitioned into smaller squares.

3. The greatest common factor of 9 and 4 is 1. The greatest common factor of 27 and 12 is 3. The greatest common factor is the same as the side length of the smallest square.

Are You Ready for More?

We have seen some examples of rectangle tilings. A *tiling* means a way to completely cover a shape with other shapes, without any gaps or overlaps. For example, here is a tiling of rectangle $KXWJ$ with 2 large squares, 3 medium squares, 1 small square, and 2 tiny squares.



Some of the squares used to tile this rectangle have the same size.

Might it be possible to tile a rectangle with squares where the squares are *all different sizes*?

If you think it is possible, find such a rectangle and such a tiling. If you think it is not possible, explain why it is not possible.

Student Response

Such tilings do exist, but they are hard to find! In fact, people thought that it was impossible for a long time. An example is a 32-by-33 rectangle that can be tiled with squares of side length 18, 15, 8, 7, 4, 14, 10, 9, and 1. (For more examples of solutions and more history on the matter, research Martin Gardner's November 1958 column in *Scientific American*.)

When presented with this problem, people usually think that there are no rectangles that can be tiled with squares that are all different sizes. Productive and enjoyable conversations can ensue from trying to explain why.

Activity Synthesis

It is not necessary for students to understand a general argument for why chopping rectangles can help you know the greatest common factor of two numbers. However, for this particular example, students may notice that:

- All of the segments in the larger partitioned rectangle are three times longer than their corresponding segments in the smaller partitioned rectangle.
- 27 and 12 are each 3 times larger than 9 and 4, respectively.

3.4 It’s All About Fractions

Optional: 30 minutes

This activity extends the work with rectangles and fractions to continued fractions. Continued fractions are not a part of grade-level work, but they can be reasoned about and rewritten using grade-level skills for operating on fractions (MP8). In particular, the insight that $\frac{1}{\frac{a}{b}} = \frac{b}{a}$ (a special case of invert and multiply) is helpful.

In this activity, students consolidate their understanding about how the greatest common factor of the numerator and denominator of a fraction can help them write an equivalent fraction whose numerator and denominator have greatest common factor 1—sometimes called “lowest terms.”

Building On

- 6.NS.B.4

Addressing

- 6.NS.A

Launch

Arrange students in groups of 2. Students work alone and compare their work with a partner.

Consider a quick warm-up like this:

Write a fraction that is equal to each expression:

1. $3 + \frac{1}{5}$
2. $\frac{1}{3+\frac{1}{5}}$
3. $2 + \frac{1}{3+\frac{1}{5}}$

Access for Students with Disabilities

Representation: Develop Language and Symbols. Eliminate barriers and provide concrete manipulatives to connect symbols to concrete objects or values. For example, provide students with access to the four large rectangles prepared on graph paper.
Supports accessibility for: Visual-spatial processing; Fine-motor skills

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Student Task Statement

1. Accurately draw a 37-by-16 rectangle. (Use graph paper, if possible.)

a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. How many squares of each size are there?

c. What are the dimensions of the last square you drew?

d. What does this have to do with $2 + \frac{1}{3+\frac{1}{5}}$?

2. Consider a 52-by-15 rectangle.

a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. Write a fraction equal to this expression: $3 + \frac{1}{2+\frac{1}{7}}$.

c. Notice some connections between the rectangle and the fraction.

d. What is the greatest common factor of 52 and 15?

3. Consider a 98-by-21 rectangle.

a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. Write a fraction equal to this expression: $4 + \frac{1}{1+\frac{7}{14}}$.

c. Notice some connections between the rectangle and the fraction.

d. What is the greatest common factor of 98 and 21?

4. Consider a 121-by-38 rectangle.

a. Use the decomposition-into-squares process to write a continued fraction for $\frac{121}{38}$. Verify that it works.

b. What is the greatest common factor of 121 and 38?

Student Response

1. 37-by-16 rectangle
    a. Diagrams vary.

    b. Two 16-by-16 squares, three 5-by-5 squares, five 1-by-1 squares

    c. 1

    d. 2, 3, and 5 are numbers that appear in the continued fraction.

2. 52-by-15 rectangle
    a. Diagrams vary.

    b. $\frac{52}{15}$

    c. When the continued fraction is rewritten, the numerator and denominator equal the side lengths of the rectangle. When the rectangle was partitioned into squares, there were 3, 2, and 7 squares of different sizes. These match the numbers in the continued fraction that was given.

    d. 1

3. 98-by-21 rectangle
    a. Diagrams vary.

    b. $\frac{98}{21}$ or $\frac{14}{3}$

    c. The side lengths of the rectangle were 98 and 21, and the fraction is $\frac{98}{21}$ (or equivalent). The fraction written with the smallest possible numbers is $\frac{14}{3}$, and both 14 and 3 are multiplied by 7, the result is $\frac{98}{21}$.

    d. 7

4. 121-by-38 rectangle
    a. $3+\frac{1}{5+\frac{1}{2+\frac{1}{3}}}$

    b. 1